Chapter 1

Functions and Limits

1.1 Introduction:

The temperature at which water boils depends on the elevation above sea level. The boiling point drops as you ascend. Also interest paid on a cash investment depends on the length of time for which the investment is held. In both the cases the value of one variable quantity which might be y, depends on the another variable quantity which we might be x. Since value of y is completely determined by the value of x, we say that y is a function of x.

1.2 Basic concepts :

1-Quantity :

Anything that can be measured or counted is called a quantity . length, mass, volume , time , age are examples of quantities. These are of two types :

i -Constants :

A constant is a quantity which retains the same values throughout a problem .i.e., the quantity to which we can not assign any other value. e.g., the sum of angles of a triangle is a constant.

ii -Variables :

Those quantities which go on changing during a given interval of time and consequently assume an unlimited numbers of different values.

Kinds of variables :

(a) Independent variables :

The variables which varies independently. i.e., the variable in which the changes(in values) produce corresponding changes in other variable.

(b) Dependent variables :

That variable whose value depends upon the value assigned to the other variable. e.g., volume of a sphere is given by $V = \frac{4}{3}\pi r^3$ varies only when r (radius) varies. So , r is an independent variable. Where V is dependent variable .

1.3 Function :

If two variables have some relation with each other, then dependent variable is called function of independent variable. i.e.,

If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y, then we say that *"y is a function of x"*

The usual notation to express that one variable y is a function of another variable x is.

$y = f(x)$ or $y = g(x)$ etc.

A function with two or more independent variables is represented by the symbol f(x,y) or g(x, y, z) etc.

Example I:

Area of a circle is given as $A = \pi r^2$

Since $A = \pi r^2$ varies as r varies ,so Area of the circle is a function of 'r'.

The equation $A = \pi r^2$ is a rule that tells how to calculate unique output of A for each possible input value of radius r.

Note: i. The set of all possible input values of the radius is called the ***domain of the function.***

ii. The set of all out put values of the area is the ***range of the functions.***

Example 2:

Let $y = x^2 + 3x + 2$ be an equation. Since for each value of x there is always a corresponding value of y, therefore y is said to be a function of x.

There are some examples of function.

$y = \text{six}^2\, x$

$$y = \frac{\ln x}{x}$$

$y = x\,e^x$

1.4 Evaluation of function :

A function y = f(x) can be evaluated by substituting that particular value in place of x in the given function.

Example 3:

Suppose that the function f is defined for all real numbers x by the formula $f(x) = 2(x-1) + 3$

Evaluate f at the input value, 0, 2, x + 2 and f(f(2))

Solution:

In each case we substitute the given input value for x into the formula for f.

$f(0) = 2(0-1) + 3 = -2 + 3 = 1$
$f(2) = 2(2-1) + 3 = 2 + 3 = 5$
$f(x+2) = 2(x+2-1) + 3 = 2x + 5$
$f(f(2)) = 2(f(2)-1) + 3 = 2(5-1) + 3 = 11$

Example 4:

If $f(x) = 2x^2 + 4x + 9$, find the value of $\dfrac{f(3) - f(1)}{f(-1) + f(0)}$

Solution : Since $f(x) = 2x^2 + 4x + 9$

$f(0) = 2(0) + 4(0) + 9 = 9$

$$f(1) = 2(1)^2 + 4(1) + 9 = 15$$
$$f(-1) = 2(-1)^2 + 4(-1) + 9 = 7$$
$$f(3) = 2(3)^2 + 4(3) + 9 = 39$$
$$\therefore \quad \frac{f(3) - f(1)}{f(-1) + f(0)} = \frac{39-15}{7+9} = \frac{24}{16} = \frac{3}{2}$$

1.5 Algebra of Function:

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions.

If f and g are function then for every value of x that belongs to the domains for both f and g, we define functions f + g, f – g and fg as

$$(f+g)(x) = f(g) + g(x)$$
$$(f-g)(x) = f(x) - g(f)$$
$$(fg)(x) = f(x) \cdot g(x)$$

At the point at which $g(x) \neq 0$, we can, define the function f/g by the formula.

$$\left[\frac{f}{g}\right](x) = \frac{f(x)}{g(x)} \text{ where } g(x) \neq 0$$

Function can also be multiplied by a real constant say c. Then function cf is defined for all x in the domain of f by.

$$f(c\,x) = c\,f(x)$$

Example 5:

If f(x) = In x, Prove that

i. $f(p\,q) = f(p) + f(q)$ ii. $F\left(\frac{p}{q}\right) = f(p) - f(q)$

Solution:

$$f(x) = \text{In } x$$

i.
$$\begin{aligned} f(p\,q) &= \text{In}(p\,q) \\ &= \text{In } p + \text{In } q \\ &= f(p) + f(q) \end{aligned}$$

ii.
$$\begin{aligned} f\left[\frac{p}{q}\right] &= \text{In}\left[\frac{p}{q}\right] \\ &= \text{In } p - \text{In } q \\ &= f(p) - f(q) \end{aligned}$$

Example 6:

If f(x) = Sin x + Cos x, Show that $f(x + \pi) = -f(x)$

Solution:

Since $f(x) = \text{Sin } x + \text{Cos } x$

$$f(x+\pi) = \text{Sin}(x+\pi) + \text{Cos}(x+\pi)$$
$$= \sin x \cos \pi + \cos x \sin \pi + \cos x \cos \pi - \sin x \sin \pi$$
$$= \text{Sin } x\,(-1) + \text{Cos } x\,(0) + \text{Cos } x\,(-1) - \text{Sin } x\,(0)$$
$$= -\text{Sin } x - \text{Cos } x$$
$$= -(\text{Sin } x + \text{Cos } x)$$
$$f(x+\pi) = -f(x)$$

1.6 Types of Functions and Their Classification

1.6.1 Explicit and Implicit Functions

Explicit Functions :

A function which is given in terms of the independent variable is called an Explicit function.

It can be expressed as $y = f(x)$,
where y is dependent variable and x is an independent variable. i.e.,

i $y = x^2 + 3x - 8$

ii. $y = \sqrt{x^2 + 2x}$

iii. $y = 1 + \sin 2x$

Implicit Functions :

A function which can be written in terms of both dependent and independent variable is called an Implicit function.

It can be expressed as $f(x, y) = 0$

i.e., i. $x^3 + 3x^2 + 4xy^2 + y^3 + 3 = 0$

ii. $x^2 \sin y + y \cos x = 7$

iii. $\sin xy + y^2 = x$

Note : Some implicit functions can be reduced in Explicit functions e.g.

(i) $x^2 y + x y = a^3$ can be reduced as $y = \dfrac{a^3}{x^2 + x}$

(ii) $y - xy - x^2 + 4 = 0$ can be reduced as $y = \dfrac{x^2 - 4}{1 - x}$

1.6.2 Algebraic and Transcendental Functions

Algebraic functions are functions $y = f(x)$ satisfying an equation of the form $p_o(x)\, y^n + p_1(x) y^{n-1} + \ldots\ldots + p_{n-1}(x)\, y + p_n(x) = 0$
Where $p_o(x), p_1(x), \ldots\ldots, p_n(x)$ are polynomial in x

i.e., i. $y = x^3 + 3x^2 + \dfrac{2}{x} + \dfrac{4}{x^2} + 5$

ii. $y = a x^2 + b x + c$

are examples of **algebraic functions.**

On the other hand **transcendental functions** are the functions which are not algebraic.

i.e.
i. Trigonometric functions
$\sin x, \cos x, \tan x, \cot x\ \sec x$ and $\operatorname{cosec} x$
ii. Inverse trigonometric functions
$\sin^{-1}x, \cos^{-1}x, \tan^{-1}x, \cot^{-1}x, \sec^{-1}x$ & $\operatorname{cosec}^{-1}x$
iii. Logarithmic functions
$\log x$ or $\ln x$
iv. Exponential functions
a^x and e^x

are examples of transcendental functions.

1.6.3 Even and Odd Functions

Even function : A function f(x) is said to be an **Even function** of x

if for $x = -x \Rightarrow f(-x) = f(x)$

For example $f(x) = x^2$ and $f(x) = \cos x$ are even functions of x

As $f(-x) = (-x)^2 = x^2 = f(x)$

and $f(-x) = \cos(-x) = \cos x = f(x)$

Odd function : A function f(x) is said to be an **Odd function** of x

if for $x = -x \Rightarrow f(-x) = -f(x)$

For example $f(x) = x^3$ and $f(x) = \sin x$ are odd functions of x

As $f(-x) = (-x)^3 = -x^3 = -f(x)$

and $f(-x) = \sin(-x) = -\sin x = -f(x)$

Example 7:

Determine which functions are even, odd or neither.

a. $f(x) = 2x^3 - 9x$
b. $f(x) = 2x^4 - 3x^2 - 1$
c. $f(x) = 4x^2 - 7x + 6$

Solution:

a. $\therefore$ $f(-x) = 2(-x)^3 - 9(-x)$
$= -2x^3 + 9x$
$= -(2x^3 - 9x)$
$= -f(x)$

So f is odd

b. $\therefore$ $f(-x) = 2(-x)^4 - 3(-x)^2 - 1$
$= 2x^4 - 3x^2 - 1$
$= f(x)$

So f is even

c. $\therefore$ $f(-x) = 4\,f(-x)^2 - 7(-x) + 6$
$= 4x^2 + 7x + 6 \neq f(x)$
$\neq -f(x)$

The result is neither f(x) nor –f(x) for all values of x . So f is neither even nor odd.

Note: cos x , sec x are Even functions of x and
sin x , tan x , cot x , cosec x are Odd functions of x

Exercise 1.1

Q.1: (i) If $f(x) = \dfrac{x^2 - 3}{x + 4}$, find f(–3) (ii) If $f(x) = 3x^2 - 5x + 7$, find f(4)

Q.2: Find the value of $f\left(\dfrac{1}{x}\right)$, if $f(x) = \dfrac{1}{x^2 + 4}$

Q.3: If $f(x) = 3x^3 + 2x^2 - x + 4$, prove that $2f(3) = 25f(1)$

Q.4 : Given $f(x) = 3x^3 - ax^2 + bx + 1$, if $f(2) = -3$ and $f(-1) = 0$, Find the values of a and b

Q.5 : (a) If $f(x) = a^x$, show that

(i) $f(-p) = \dfrac{1}{f(p)}$ (ii) $f(x + y) = f(x)\,f(y)$

(b) If $f(x) = \log x$, prove that $f(x^a) = a\,f(x)$

Q.6: If $f(p) = p + \dfrac{1}{p}$. Prove the following results.

(i) $f(-p) = -f(p)$ (ii) $f\left(\dfrac{1}{p}\right) = f(p)$

Q.7: If f(x) = tan x show that $f(x + y) = \dfrac{f(x) + f(y)}{1 - f(x)\,f(y)}$

Q.8: If $f(t) = \dfrac{t^4 + t^2 + 1}{t^2}$, show that $f\left(\dfrac{1}{t}\right) = f(t)$

Q.9: If $f(x) = \dfrac{x - 1}{x + 1}$, show that $\dfrac{f(x) - f(y)}{1 + f(x)\,f(y)} = \dfrac{x - y}{1 + xy}$.

Q.10: Prove that f[f(x)] = x for the function $f(x) = \dfrac{x + 1}{x - 1}$

Q.11: Are the following functions even odd or neither?

(i) $f(x) = 4x^3 - 2x + 6$ (ii) $f(x) = \dfrac{x}{x^2 + 1}$

(iii) $f(x) = 2x^4 - 3x^2 - 1$ (iv) $f(x) = x\sqrt{x^2 - 1}$

Q.12: (i) Show that $\dfrac{e^x + 1}{e^x - 1}$ is an odd function of x.

(ii) show that $x \cdot \dfrac{a^x + 1}{a^x - 1}$ is an Even function of x

Q.13 : If $f(x) = \log \dfrac{1 - x}{1 + x}$ prove that $f(x) + f(y) = f\left(\dfrac{x + y}{1 + xy}\right)$

Answers

Q.1(i) 6 (ii) 35 **Q.2** $\dfrac{x^2}{1+4x^2}$ **Q.4** a = - 2 , b = - 6

Q.11 (i) Neither (ii) Odd (iii) Even (iv) Odd

LIMITS

1.7 Introduction:

The concept of limit of a function is one of the fundamental ideas that distinguishes calculus from algebra and trigonometry. The concept of limit is the foundation of almost all the mathematical analysis. A good understanding of limit will help to explain many theories of calculus.

1.8 Value and Limit of a Function:

There are two ways of studying a function. One is to find the value of y, the dependent variable for various values of x, the independent variable by actual substitution in the functional relation between the two variables. Another method to study the behavior of the dependent variable in the neighborhood of various values of the independent variable. The former method is finding the ***value of the function*** and the later method, which gives us more detailed information about the function than the former, is finding the ***limit of the function.***

1.8.1 Limit of a Variable: ($x \to a$)

If a variable `x' approaches a constant `a' such that the absolute value of the difference x - a becomes less than 'ϵ', any +ve number, however small, then `a' is called the limit of x and is symbolically written as $x \to a$ and read as "x approaches a" or " x tends to a".

For Example:

Suppose that x has the values $a + \dfrac{1}{10}$, $a + \dfrac{1}{10^2}$, ….., $a + \dfrac{1}{10^n}$

Since the values $\dfrac{1}{10}, \dfrac{1}{10^2}, \ldots, \dfrac{1}{10^n} \to 0$ when $n \to \infty$

Hence $x \to a$