Exercise : 3.8

Q.1 Check that y is solution of d.Eq :

(i) $x\dfrac{dy}{dx} = 1+y$, $y = cx-1$

If $y = cx-1$ is the solution of differential Eq so must satisfy it.

$$x\frac{d}{dx}[cx-1] = 1+(cx-1)$$

$$x(c\cdot 1-0) = 1+cx-1$$

$$xc = cx$$

$\because$ LHS = RHS

So $y = cx-1$ is the solution of d.Eq.

(ii) $x^2(2y+1)\dfrac{dy}{dx} - 1 = 0$ —— ①

$y^2+y = c-\frac{1}{x}$ —— ②

Diff. ② w.r.t x we have

$$2y\frac{dy}{dx} + \frac{dy}{dx} = 0+\frac{1}{x^2}$$

$$(2y+1)\frac{dy}{dx} = \frac{1}{x^2}$$

$$x^2(2y+1)\frac{dy}{dx} = 1$$

Putting in ①, we have

$$1-1 = 0$$

$$0 = 0$$

LHS = RHS

so ② is the solution of ①

Putting in ① $\dfrac{1+y^2}{e^{-x}} = \dfrac{1+y^2}{e^{-x}}$

LHS = RHS so
② represents the solution of ①

(iii) $y\dfrac{dy}{dx} - e^{2x} = 1$ —— ①

$y^2 = e^{2x}+2x+c$ —— ②

Diff. ② w.r.t. x

$$2y\frac{dy}{dx} = 2e^{2x}+2$$

$$y\frac{dy}{dx} = e^{2x}+1$$

Putting in ①, we have

$$e^{2x}+1-e^{2x} = 1$$

$$1 = 1$$

LHS = RHS

So ② represents the solution of ①

(iv) $\dfrac{1}{x}\dfrac{dy}{dx} - 2y = 0$ —— ①

$y = ce^{x^2}$ —— ②

Diff. ② w.r.t. x

$$\frac{dy}{dx} = ce^{x^2}2x$$

$$\frac{dy}{dx} = 2xy$$

Putting in ①, we have

$$\frac{1}{\cancel{x}}(2\cancel{x}y) - 2y = 0$$

$$2y-2y = 0 \Rightarrow 0 = 0$$

LHS = RHS.

So ② represents the solution of ①

(v) $\dfrac{dy}{dx} = \dfrac{y^2+1}{e^{-x}}$ —— ① $y = \tan(e^x+c)$ —— ②

Diff. ② w.r.t. x $\dfrac{dy}{dx} = \sec^2(e^x+c)\,e^x$

$$\frac{dy}{dx} = \frac{1+\tan^2(e^x+c)}{e^{-x}} \quad \because y = \tan(e^x+c)$$

Q.2 $\dfrac{dy}{dx} = -y$

Separating the variables

$$\frac{dy}{y} = -dx$$

Integrating both sides

$$\int \frac{dy}{y} = -\int dx$$

$$\ln y = -x + c$$

$$\ln y + x = c \quad \text{G. Solution.}$$

Q.3 $y\,dx + x\,dy = 0$

$$y\,dx = -x\,dy$$

$$-\frac{dy}{y} = +\frac{dx}{x} \Rightarrow \frac{dy}{y} = -\frac{dx}{x}$$

Integrating both sides

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\ln y = -\ln x + \ln c$$

$$\ln y + \ln x = \ln c$$

$$\ln(xy) = \ln c$$

$$xy = c \quad \text{G. Solution.}$$

$\because \ln x + \ln y = \ln(xy)$

$\because e^{\ln x} = x$

Q.4 $\dfrac{dy}{dx} = \dfrac{1-x}{y}$

$$y\,dy = (1-x)\,dx$$

Integrating both sides

$$\int y\,dy = \int (1-x)\,dx$$

$$\frac{y^2}{2} = x - \frac{x^2}{2} + c$$

$$\frac{y^2}{2} + \frac{x^2}{2} = x + c$$

$$x^2 + y^2 = 2(x+c) \quad \text{G. Solution.}$$

Q.5 $\dfrac{dy}{dx} = \dfrac{y}{x^2}$

$$\frac{dy}{y} = \frac{dx}{x^2}$$

Integrating both sides

$$\int \frac{dy}{y} = \int x^{-2}\,dx$$

$$\ln y = \frac{x^{-1}}{-1} + c$$

$$\ln y = \frac{-1}{x} + c$$

$$\ln y + \frac{1}{x} = c \quad \text{G. Solution.}$$

Q.6 $\sin y \,\operatorname{Cosec} x \,\dfrac{dy}{dx} = 1$

$$\sin y\,dy = \frac{dx}{\operatorname{Cosec} x}$$

$$\sin y\,dy = \sin x\,dx$$

Integrating both sides

$$\int \sin y\,dy = \int \sin x\,dx$$

$$-\cos y = -\cos x + c$$

$$\cos x - \cos y = c \quad \text{G. Solution.}$$

Q.7 $x\,dy + y(x-1)\,dx = 0$

$$x\,dy = -y(x-1)\,dx$$

$$\frac{dy}{y} = \left(\frac{1-x}{x}\right) dx$$

Integrating both sides

$$\int \frac{dy}{y} = \int \frac{1}{x}\,dx - \int 1\,dx$$

$$\ln y = \ln x - x + c$$

$$\ln y - \ln x + x = c$$

$$\ln\left(\frac{y}{x}\right) + x = c \quad \text{G. Solution.}$$

Q.8 $\frac{x^2+1}{y^2+1} = \frac{x}{y}\frac{dy}{dx}$

$\frac{x^2+1}{x}dx = \frac{y^2+1}{y}dy$

Integrating both sides

$\int\left(\frac{x^2}{x}+\frac{1}{x}\right)dx = \int\left(\frac{y^2}{y}+\frac{1}{y}\right)dy$

$\int\left(x+\frac{1}{x}\right)dx = \int(y+1/y)\,dy$

$\frac{y^2}{2}+\ln y = \frac{x^2}{2}+\ln x + c$

$\frac{y^2}{2}-\frac{x^2}{2}+\ln y - \ln x = c$

$\frac{y^2-x^2}{2}+\ln(y/x) = c$ G. Solution

Q.9 $\frac{1}{x}\frac{dy}{dx} = \frac{1}{2}(1+y^2)$

$\frac{dy}{1+y^2} = \frac{1}{2}x\,dx$

Integrating both sides

$\int\frac{dy}{1+y^2} = \frac{1}{2}\int x\,dx$

$\tan^{-1}y = \frac{1}{2}\frac{x^2}{2}+c$

$\tan^{-1}y - \frac{x^2}{4} = c$ G. Solution

Q.10 $2x^2y\frac{dy}{dx} = x^2-1$

$2y\,dy = \frac{x^2-1}{x^2}dx$

Integrating both sides

$\int 2y\,dy = \int\left(\frac{x^2}{x^2}-1/x^2\right)dx$

$2\frac{y^2}{2} = \int dx - \int x^{-2}dx$

$y^2 = x - \frac{x^{-1}}{-1}+c$

$y^2 = x+\frac{1}{x}+c$ G. Solution

Q.11 $\frac{dy}{dx}+\frac{2xy}{2y+1} = x$

$\frac{dy}{dx} = x - \frac{2xy}{2y+1}$

$\frac{dy}{dx} = \frac{2yx+x-2xy}{2y+1} = \frac{x}{2y+1}$

$(2y+1)dy = x\,dx$

Integrating both sides

$\int(2y+1)dy = \int x\,dx$

$\frac{2y^2}{2}+y = \frac{x^2}{2}+c$

$y^2+y-\frac{x^2}{2} = c$ G. Solution

Q.12 $(x^2-yx^2)\frac{dy}{dx}+y^2+xy^2=0$

$x^2(1-y)\frac{dy}{dx} = -y^2-xy^2$

$x^2(1-y)\frac{dy}{dx} = -y^2(1-x)$

$\frac{(1-y)dy}{y^2} = \frac{x-1}{x^2}dx$

Integrating both sides

$\int\left(\frac{1}{y^2}-\frac{y}{y^2}\right)dy = \int\left(\frac{x}{x^2}-1/x^2\right)dx$

$\int y^{-2}dy - \int\frac{1}{y}dy = \int\frac{1}{x}dx - \int x^{-2}dx$

$\frac{y^{-1}}{-1} - \ln y = \ln x - \frac{x^{-1}}{-1}+c$

$-\frac{1}{y} - \ln y = \ln x + \frac{1}{x} + c$

$\ln x + \ln y + \frac{1}{x} + \frac{1}{y} + c = 0$

$\ln(xy)+\frac{1}{x}+1/y+c=0$ G. Solution

Q.13 $\sec^2x\tan y\,dx + \sec^2y\tan x\,dy = 0$

$-\sec^2x\tan y\,dx = \sec^2y\tan x\,dy$

$\frac{\sec^2y}{\tan y}dy = -\frac{\sec^2x\,dx}{\tan x}$

Integrating both sides

$\int\frac{\sec^2y}{\tan y}dy = -\int\frac{\sec^2x}{\tan x}dx$

$\ln|\tan y| = -\ln|\tan x|+\ln c$

$\ln(\tan y)+\ln(\tan x) = \ln c \Rightarrow \ln(\tan y\cdot\tan x)=\ln c \Rightarrow \tan x\tan y = c$ G. Solution

Q.14 $\left(y - x\frac{dy}{dx}\right) = 2\left(y^2 + \frac{dy}{dx}\right)$

$y - x\frac{dy}{dx} = 2y^2 + 2\frac{dy}{dx}$

$y - 2y^2 = 2\frac{dy}{dx} + x\frac{dy}{dx}$

$(2+x)\frac{dy}{dx} = (y - 2y^2)$

$\frac{dy}{y(1-2y)} = \frac{dx}{2+x}$

Integrating both sides.

$\frac{1}{y(1-2y)} = \frac{A}{y} + \frac{B}{1-2y}$

$1 = A(1-2y) + B(y)$

Putting $y=0$ and $1-2y=0 \Rightarrow y = 1/2$

$\boxed{1 = A}$ $\quad 1 = B\frac{1}{2} \Rightarrow \boxed{B = 2}$

$\int\left(\frac{1}{y} + \frac{2}{1-2y}\right)dy = \int\frac{dx}{x+2}$

$\int\frac{1}{y}dy - \int\frac{-2}{1-2y}dy = \int\frac{dx}{x+2}$

$\ln y - \ln(1-2y) = \ln(x+2) + \ln c$

$\ln y - \ln(1-2y) - \ln(x+2) = \ln c$

$\ln\left(\frac{y}{(x+2)(1-2y)}\right) = \ln c$

$\frac{y}{(x+2)(1-2y)} = c$

$y = (x+2)(1-2y)c$ G-Solution

Q.15 $1 + \cos x \tan y\frac{dy}{dx} = 0$

$\cos x \tan y\frac{dy}{dx} = -1$

$\tan y\, dy = \frac{-dx}{\cos x}$

$\tan y\, dy = -\sec x\, dx$

Integrating both sides.

$\int \tan y\, dy = -\int \sec x\, dx$

$\ln|\sec x| = -\ln|\sec x + \tan x| + \ln c$

$\ln(\sec x) + \ln(\sec x + \tan x) = \ln c$

$\ln|\sec x(\sec x + \tan x)| = \ln c$

$\sec x(\sec x + \tan x) = c$ G-Solution.

Q.16 $y - x\frac{dy}{dx} = 3\left(1 + x\frac{dy}{dx}\right)$

$y - x\frac{dy}{dx} = 3 + 3x\frac{dy}{dx}$

$y - 3 = (x + 3x)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{y-3}{4x}$

$\frac{dy}{y-3} = \frac{dx}{4x}$

Integrating both sides

$\int\frac{dy}{y-3} = \frac{1}{4}\int\frac{dx}{x}$

$\ln(y-3) = \frac{1}{4}\ln x + \ln c$

$\ln(y-3) - \ln x^{1/4} = \ln c$

$\ln\left((y-3)x^{1/4}\right) = \ln c$

$x^{1/4}(y-3) = c$ G-Solution

Q.17 $\sec x + \tan y\frac{dy}{dx} = 0$

$\tan y\frac{dy}{dx} = -\sec x$

$\tan y\, dy = -\sec x\, dx$

Integrating both sides

$\ln|\sec y| = -\ln(\sec x + \tan x) + \ln c$

$\ln(\sec y) + \ln(\sec x + \tan x) = \ln c$

$\ln(\sec y(\sec x + \tan x)) = \ln c$

$\sec y(\sec x + \tan x) = C$

G-Solution.

Q.18 $(e^{x}+e^{-x})\frac{dy}{dx} = e^{x}-e^{-x}$

$$\frac{dy}{dx} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

$$dy = \left(\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right)dx$$

Integrating both sides

$$\int dy = \int\left(\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right)dx$$

$$y = \ln[e^{x}+e^{-x}] + c$$ G.Solution.

Q.19 $\frac{dy}{dx} - x = xy^{2}$

$$\frac{dy}{dx} = xy^{2}+x$$

$$\frac{dy}{dx} = x(y^{2}+1)$$

$$\frac{dy}{1+y^{2}} = x\,dx$$

Integrating both sides

$$\int\frac{dy}{1+y^{2}} = \int x\,dx$$

$$\tan^{-1}y = \frac{x^{2}}{2}+c$$ G.Solution.

For particular solution, $y=1, x=0$

$$\tan^{-1}(1) = 0 + c \Rightarrow c = \pi/4$$

Thus $\tan^{-1}y = \frac{x^{2}}{2}+\frac{\pi}{4}$ P.Solution.

Q.20 $\frac{dx}{dt} = 2x$

$$\frac{dx}{x} = 2\,dt$$

Integrating both sides

$$\int\frac{dx}{x} = 2\int dt$$

$$\ln x = 2t + c$$ G.Solution

For particular solution

$x=4$ and $t=0$

$$\ln 4 = 0 + c \Rightarrow c = \ln 4$$

$$\ln x = 2t + \ln 4$$

$$\ln x - \ln 4 = 2t$$

$$\ln\left(\frac{x}{4}\right) = 2t$$ Particular Solution.

Q.21 $\frac{ds}{dt} + 2st = 0$

$$\frac{ds}{dt} = -2st$$

$$\frac{ds}{s} = -2t\,dt$$

Integrating both sides

$$\int\frac{ds}{s} = -2\int t\,dt$$

$$\ln|s| = -\frac{2t^{2}}{2}+c$$

$$\ln s = -t^{2}+c$$

$$\ln s + t^{2} = c$$ G.Solution.

For particular solution.

$s = 4e \quad t = 0$

$$\ln(4e) + 0 = c \Rightarrow c = \ln 4 + \ln e$$

$$c = \ln 4 + 1 \quad \because \ln e = 1$$

Thus Particular Solution is

$$\ln s + t^{2} = \ln 4 + 1$$

$$\ln s - \ln 4 + t^{2} = 1$$

$$\ln\left[\frac{s}{4}\right] + t^{2} = 1$$ P.Solution.

Q.22 Let N be the number of bacteria at present then

$$\frac{dN}{dt} \propto N$$

$$\frac{dN}{dt} = rN \quad (r \text{ is Constant})$$

$$\frac{dN}{N} = r\,dt$$

Integrating both sides

$$\int \frac{dN}{N} = r\int dt$$

$$\ln|N| = rt + c$$

$$N = e^{rt+c}$$

$$N = e^{rt} \cdot e^{c}$$

Applying initial Conditions.

$t = 0 \quad N = 200$

$$200 = e^{c} \cdot e^{0} \Rightarrow e^{c} = 200$$

$$N = 200\,e^{rt}$$

when $t = 2$ hour $N = 400$

$$400 = 200\,e^{r(2)}$$

$$e^{2r} = \frac{400}{200} = 2$$

$$2r \ln e = \ln(2)$$

$$r = \frac{1}{2}\ln(2)$$

$$r = \ln(2)^{1/2}$$

$N = ?$ when $t = 4$

$$N = 200\,e^{4\left(\frac{1}{2}\ln 2\right)}$$

$$N = 200\,e^{2\ln 2}$$

$$N = 200\,e^{\ln 2^2} = 200\,e^{\ln 4}$$

$$N = 200 \times 4 = 800 \text{ Ans} \qquad \because e^{\ln} = 1$$

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Q.23 $V = 2450$ cm/sec

By the definition

$$\frac{dV}{dt} = -g$$

because body is thrown upward against gravitational field

$$dV = -g\,dt$$

Integrating both sides

$$\int dV = -g\int dt$$

$$V = -gt + c_1$$

Applying initial Conditions.

$V = 2450 \qquad t = 0$

$$c_1 = 2450 \qquad \because g = 980\ \text{cm/sec}^2$$

$$\Rightarrow \boxed{V = 2450 - 980t}$$

Also $V = \frac{dh}{dt} \qquad \because S = h$

$$dh = V\,dt$$

Integrating both sides

$$\int dh = \int (2450 - 980t)\,dt$$

$$h = 2450t - \frac{980t^2}{2} + c_2$$

$c_2 = 0$ as $h = 0$ at $t = 0$ $\quad \because V = 2450 - 980t$

$$\Rightarrow \boxed{h = 2450t - 490t^2}$$

<u>Maximum height:-</u>

For maximum height

$$\frac{dh}{dt} = 0 \Rightarrow 2450 - 980t = 0$$

$$t = \frac{2450}{980} = 2.5$$

$$\frac{d^2h}{dt^2} = 0 - 980 < 0$$

h is maximum at $t = 2.5$

$$h(\max) = 2450(2.5) - 490(2.5)^2$$

$$h(\max) = 6125 - 3062.5 = \boxed{3062.5\ \text{cm}}$$

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