Integration By Parts:-

If $f(x)$ and $g(x)$ are two functions then their by parts integration is defined as

$$\int \underset{I}{f(x)} \cdot \underset{II}{g(x)} = f(x)\int g(x)dx - \int\left[\left(\int g(x)dx\right)\left(\frac{d}{dx}f(x)\right)\right]dx$$

where I and II are selected by the nature of function that which is integralable easily and which is differentiable easily.

Exercise 3.4

Q.1/ Evaluate the followings by part integral

(i) $\int \underset{I}{x} \underset{II}{\sin x}\, dx$

$= x(-\cos x) - \int(-\cos x)\frac{d}{dx}(x)\,dx$

$= -x\cos x + \int \cos x\, dx$

$= -x\cos x + \sin x + c$ Ans.

(ii) $\int \ln x\, dx$

$= \int \underset{II}{1} \cdot \underset{I}{\ln x}\, dx$

$= \ln x(x) - \int x \cdot \frac{d}{dx}(\ln x)\,dx$

$= x\ln x - \int x \cdot \frac{1}{x}\,dx$

$= x\ln x - \int dx$

$= x\ln x - x + c$ Ans.

(iii) $\int \underset{II}{x} \underset{I}{\ln x}\, dx$

$= (\ln x)\left(\frac{x^2}{2}\right) - \int \frac{x^2}{2} \cdot \frac{d}{dx}(\ln x)\,dx$

$= \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\, dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) + c$

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$ Ans.

(iv) $\int \underset{II}{x^2} \underset{I}{\ln x}\, dx$

$= \ln x\left(\frac{x^3}{3}\right) - \int \frac{x^3}{3}\frac{d}{dx}(\ln x)\,dx$

$= \frac{x^3}{3}\ln x - \int \frac{x^3}{3} \cdot \frac{1}{x}\,dx$

$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\, dx$

$= \frac{x^3}{3}\ln x - \frac{1}{3}\left(\frac{x^3}{3}\right) + c$

$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + c$ Ans.

(v) $\int \underset{II}{x^3} \underset{I}{\ln x}\, dx$

$= \ln x\left(\frac{x^4}{4}\right) - \int \frac{x^4}{4}\frac{d}{dx}(\ln x)\,dx$

$= \frac{x^4}{4}\ln x - \int \frac{x^4}{4} \cdot \frac{1}{x}\,dx$

$= \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3\, dx$

$= \frac{x^4}{4}\ln x - \frac{1}{4}\left(\frac{x^4}{4}\right) + c$

$= \frac{x^4}{4}\ln x - \frac{x^4}{16} + c$ Ans.

(vi) $\int \underset{II}{x^4} \underset{I}{\ln x}\, dx$

$= \ln x\left(\frac{x^5}{5}\right) - \int \frac{x^5}{5} \cdot \frac{d}{dx}(\ln x)\,dx$

$= \frac{x^5}{5}\ln x - \int \frac{x^5}{5} \cdot \frac{1}{x}\,dx$

$= \frac{x^5}{5}\ln x - \frac{1}{5}\int x^4\, dx$

$= \frac{x^5}{5}\ln x - \frac{1}{5}\left(\frac{x^5}{5}\right) + c$

$= \frac{x^5}{5}\ln x - \frac{x^5}{25} + c$ Ans

(vii) $\int \tan^{-1} x \, dx$

$= \int \underset{II}{1} \cdot \underset{I}{\tan^{-1} x} \, dx$

$= \tan^{-1} x \cdot (x) - \int x \cdot \frac{d}{dx}(\tan^{-1} x) dx$

$= x \tan^{-1} x - \int x \cdot \frac{1}{1+x^2} \, dx$

$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$

$= x \tan^{-1} x - \frac{1}{2} \ln[1+x^2] + c$ Ans.

(viii) $\int \underset{I}{x^2} \underset{II}{\sin x} \, dx$

$= x^2(-\cos x) - \int (-\cos x) \frac{d}{dx}(x^2) dx$

$= -x^2 \cos x + \int \cos x \, 2x \, dx$

$= -x^2 \cos x + 2 \int \underset{II}{x} \underset{I}{\cos x} \, dx$

$= -x^2 \cos x + 2 [x (\sin x) - \int \sin x \cdot \frac{d}{dx} x \, dx]$

$= -x^2 \cos x + 2 [x \sin x - \int \sin x \, dx]$

$= -x^2 \cos x + 2 [x \sin x + \cos x] + c$

$= -x^2 \cos x + 2x \sin x + 2 \cos x + c$ Ans.

(ix) $\int \underset{II}{x^2} \underset{I}{\tan^{-1} x} \, dx$

$= \tan^{-1} x \left(\frac{x^3}{3}\right) - \int \frac{x^3}{3} \frac{d}{dx} (\tan^{-1} x) dx$

$= \frac{x^3}{3} \tan^{-1} x - \int \frac{x^3}{3} \cdot \frac{1}{1+x^2} \, dx$

$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} \, dx$

$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \left(x - \frac{x}{1+x^2}\right) dx$

$\left\{ \because \frac{x^3}{1+x^2} = \left(x - \frac{x}{1+x^2}\right) \right\}$

$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int x \, dx + \frac{1}{3} \int \frac{x}{1+x^2} \, dx$

$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \left(\frac{x^2}{2}\right) + \frac{1}{6} \int \frac{2x}{1+x^2} dx$

$= \frac{x^3}{3} \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \ln(x^2+1) + c$ Ans.

(x) $\int \underset{II}{x} \underset{I}{\tan^{-1} x} \, dx$

$= \tan^{-1} x \left(\frac{x^2}{2}\right) - \int \frac{x^2}{2} \cdot \frac{d}{dx} (\tan^{-1} x) dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int x^2 \cdot \frac{1}{1+x^2} \, dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} \, dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2+1-1}{x^2+1} \, dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\right) dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2+1}{x^2+1} dx + \frac{1}{2} \int \frac{dx}{1+x^2}$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{dx}{1+x^2}$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + c$ Ans.

(xi) $\int \underset{II}{x^3} \underset{I}{\tan^{-1} x} \, dx$

$= \tan^{-1} x \left(\frac{x^4}{4}\right) - \int \frac{x^4}{4} \cdot \frac{d}{dx} (\tan^{-1} x) dx$

$= \frac{x^4}{4} \tan^{-1} x - \int \frac{x^4}{4} \cdot \frac{1}{1+x^2} \, dx$

$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int \frac{x^4}{1+x^2} \, dx$

$\left\{ \because \frac{x^4}{1+x^2} = x^2 - 1 + \frac{1}{1+x^2} \right\}$

$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int \left(x^2 - 1 + \frac{1}{1+x^2}\right) dx$

$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \int x^2 dx + \frac{1}{4} \int dx - \frac{1}{4} \int \frac{1 dx}{1+x^2}$

$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \left(\frac{x^3}{3}\right) + \frac{1}{4} x - \frac{1}{4} \tan^{-1} x + c$

$= \frac{x^4}{4} \tan^{-1} x - \frac{x^3}{12} + \frac{x}{4} - \frac{1}{4} \tan^{-1} x + c$

Ans.

(xii) $\int \underset{I}{x^3} \underset{II}{\cos x}\, dx$

$$= x^3(\sin x) - \int \sin x \frac{d}{dx}(x^3)\, dx$$
$$= x^3 \sin x - \int \sin x \cdot 3x^2\, dx$$
$$= x^3 \sin x - 3\int \underset{I}{x^2} \underset{II}{\sin x}\, dx$$
$$= x^3 \sin x - 3\left[x^2(-\cos x) - \int(-\cos x)\frac{d}{dx}(x^2)\, dx\right]$$
$$= x^3 \sin x - 3\left[-x^2\cos x + \int 2x\cos x\, dx\right]$$
$$= x^3 \sin x + 3x^2\cos x - 3\int 2x\cos x\, dx$$
$$= x^3 \sin x + 3x^2\cos x - 6\int \underset{I}{x} \underset{II}{\cos x}\, dx$$
$$= x^3 \sin x + 3x^2\cos x - 6\left[x\sin x - \int \sin x \cdot \frac{dx}{dx}\, dx\right]$$
$$= x^3 \sin x + 3x^2\cos x - 6\left[x\sin x - \int \sin x\, dx\right]$$
$$= x^3 \sin x + 3x^2\cos x - 6x\sin x + 6\int \sin x\, dx$$
$$= x^3 \sin x + 3x^2\cos x - 6x\sin x + 6(-\cos x) + c$$
$$= x^3 \sin x + 3x^2\cos x - 6x\sin x - 6\cos x + c \quad \text{Ans.}$$

(xiii) $\int \sin^{-1}x\, dx$

$$= \int \underset{II}{1} \cdot \underset{I}{\sin^{-1}x}\, dx$$
$$= \sin^{-1}x\,(x) - \int x \frac{d}{dx}(\sin^{-1}x)\, dx$$
$$= x\sin^{-1}x - \int x \cdot \frac{1}{\sqrt{1-x^2}}\, dx$$
$$= x\sin^{-1}x - \int (1-x^2)^{-1/2}\, x\, dx$$
$$= x\sin^{-1}x + \frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\, dx$$
$$= x\sin^{-1}x + \frac{\cancel{1}}{\cancel{2}} \frac{(1-x^2)^{1/2}}{\cancel{1/2}} + c$$
$$= x\sin^{-1}x + \sqrt{1-x^2} + c \quad \text{Ans.}$$

(xiv) $\int \underset{II}{x}\, \underset{I}{\sin^{-1}x}\, dx$

$$= \sin^{-1}x\left(\frac{x^2}{2}\right) - \int \frac{x^2}{2}\frac{d}{dx}(\sin^{-1}x)\, dx$$
$$= \frac{x^2}{2}\sin^{-1}x - \int \frac{x^2}{2\sqrt{1-x^2}}\, dx$$
$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \frac{-x^2}{\sqrt{1-x^2}}\, dx$$
$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}}\, dx$$
$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int\left(\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}\right) dx$$
$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \frac{1-x^2}{\sqrt{1-x^2}}\, dx - \frac{1}{2}\int \frac{dx}{\sqrt{1-x^2}}$$
$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \sqrt{1-x^2}\, dx - \frac{1}{2}\sin^{-1}x + c$$
$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right] - \frac{1}{2}\sin^{-1}x + c$$
$$= \frac{x^2}{2}\sin^{-1}x + \frac{x\sqrt{1-x^2}}{4} + \frac{1}{4}\sin^{-1}x - \frac{1}{2}\sin^{-1}x + c$$
$$= \frac{x^2}{2}\sin^{-1}x + \frac{x\sqrt{1-x^2}}{4} + \left(\frac{1}{4} - \frac{1}{2}\right)\sin^{-1}x + c$$
$$= \frac{x^2}{2}\sin^{-1}x + \frac{x\sqrt{1-x^2}}{4} + \left(\frac{1-2}{4}\right)\sin^{-1}x + c$$
$$= \frac{x^2}{2}\sin^{-1}x + \frac{x\sqrt{1-x^2}}{4} - \frac{1}{4}\sin^{-1}x + c \quad \text{Ans.}$$

(xv) $\int e^x(\sin x\cos x)\, dx$

$$= \frac{1}{2}\int e^x(2\sin x\cos x)\, dx$$
$$= \frac{1}{2}\int e^x \sin 2x\, dx$$

Let $I = \int \underset{I}{e^x}\, \underset{II}{\sin 2x}$

$$I = e^x\left[-\frac{\cos 2x}{2}\right] - \int\left(-\frac{\cos 2x}{2}\right)\frac{d}{dx}(e^x)\, dx$$
$$I = -\frac{e^x\cos 2x}{2} + \frac{1}{2}\int \underset{I}{e^x}\, \underset{II}{\cos 2x}\, dx$$
$$I = -\frac{e^x\cos 2x}{2} + \frac{1}{2}\left[\frac{e^x\sin 2x}{2} - \int \frac{\sin 2x}{2}\frac{d}{dx}(e^x)\, dx\right]$$
$$I = -\frac{e^x\cos 2x}{2} + \frac{1}{4}e^x\sin 2x - \frac{1}{4}\int e^x \sin 2x\, dx$$

$$I = -\frac{e^x \cos 2x}{2} + \frac{e^x \sin 2x}{4} - \frac{1}{4} I$$

$$I + \frac{1}{4} I = -\frac{e^x \cos 2x}{2} + \frac{e^x \sin 2x}{4}$$

$$\left(1+\frac{1}{4}\right) I = -\frac{e^x \cos 2x}{2} + \frac{e^x \sin 2x}{4}$$

$$\frac{5}{4} I = -\frac{e^x \cos 2x}{2} + \frac{e^x \sin 2x}{4}$$

$$I = \frac{4}{5}\left[\frac{e^x \sin 2x}{4} - \frac{e^x \cos 2x}{2}\right]$$

$$\Rightarrow \frac{1}{2}\int e^x \sin 2x \, dx$$

$$= \frac{1}{2} \times \frac{4}{5}\left[\frac{e^x \sin 2x}{4} - \frac{e^x \cos 2x}{2}\right] + c$$

$$= \frac{2}{5}\left[\frac{e^x \sin 2x}{4} - \frac{e^x \cos 2x}{2}\right] + c$$

$$= \frac{2}{5}\,\frac{e^x \sin 2x}{4} - \frac{2}{5}\cdot\frac{e^x \cos 2x}{2} + c$$

$$= \frac{1}{10} e^x \sin 2x - \frac{1}{5} e^x \cos 2x + c \quad \text{Ans.}$$

(xvi) $\int x \sin x \cos x \, dx$

$$= \frac{1}{2}\int x \,(2 \sin x \cos x)\, dx$$

$$= \frac{1}{2}\int \underset{I}{x} \, \underset{II}{\sin 2x} \, dx$$

$$= \frac{1}{2}\left[x\left(-\frac{\cos 2x}{2}\right) - \int\left(-\frac{\cos 2x}{2}\right)\frac{d(x)}{dx}\, dx\right]$$

$$= \frac{1}{2}\left[-\frac{x \cos 2x}{2} + \int \frac{\cos 2x}{2} dx\right]$$

$$= -\frac{x \cos 2x}{4} + \frac{1}{4}\int \cos 2x \, dx$$

$$= -\frac{x \cos 2x}{4} + \frac{1}{4}\left(\frac{\sin 2x}{2}\right) + c$$

$$= -\frac{x \cos 2x}{4} + \frac{\sin 2x}{8} + c \quad \text{Ans.}$$

(xvii) $\int x \cos^2 x \, dx$ $\qquad \cos^2 x = \frac{1+\cos 2x}{2}$

$$= \int x\left(\frac{1+\cos 2x}{2}\right) dx \quad \therefore$$

$$= \int\left(\frac{x}{2} + \frac{x \cos 2x}{2}\right) dx$$

$$= \int \frac{x}{2} dx + \int \frac{x \cos 2x}{2} dx$$

$$= \frac{x^2}{2\cdot 2} + \frac{1}{2}\int \underset{I}{x} \, \underset{II}{\cos 2x} \, dx$$

$$= \frac{x^2}{4} + \frac{1}{2}\left[x\left(\frac{\sin 2x}{2}\right) - \int \frac{\sin 2x}{2}\cdot\frac{d(x)}{dx}\, dx\right]$$

$$= \frac{x^2}{4} + \frac{1}{2}\left[\frac{x \sin 2x}{2} - \frac{1}{2}\int \sin 2x \, dx\right]$$

$$= \frac{x^2}{4} + \frac{1}{2}\left[\frac{x \sin 2x}{2} - \frac{1}{2}\left(-\frac{\cos 2x}{2}\right)\right] + c$$

$$= \frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + c \quad \text{Ans.}$$

(xviii) $\int x \sin^2 x \, dx$

$$= \int x\left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\int x \, dx - \frac{1}{2}\int \underset{I}{x} \, \underset{II}{\cos 2x} \, dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right) - \frac{1}{2}\left[x\left(\frac{\sin 2x}{2}\right) - \int \frac{\sin 2x}{2}\frac{d(x)}{dx}\, dx\right]$$

$$= \frac{x^2}{4} - \frac{1}{2}\left[\frac{x \sin 2x}{2} - \frac{1}{2}\int \sin 2x \, dx\right]$$

$$= \frac{x^2}{4} - \frac{1}{2}\left[\frac{x \sin 2x}{2} - \frac{1}{2}\left(-\frac{\cos 2x}{2}\right)\right] + c$$

$$= \frac{x^2}{4} - \frac{x \sin 2x}{4} - \frac{\cos 2x}{8} + c \quad \text{Ans.}$$

(xix) $\int (\ln x)^2 \, dx$

$$= \int \underset{II}{1} \cdot \underset{I}{(\ln x)^2} \, dx$$

$$= (\ln x)^2 (x) - \int x \, \frac{d}{dx}\left((\ln x)^2\right) dx$$

$$= x(\ln x)^2 - \int x \cdot 2(\ln x)\cdot \frac{1}{x} \, dx$$

$$= x(\ln x)^2 - 2\int \ln x \, dx$$

$$= x(\ln x)^2 - 2\int \underset{II}{1} \cdot \underset{II}{(\ln x)} \, dx$$

$$= x(\ln x)^2 - 2\left[x \cdot \ln x - \int x \cdot \frac{d}{dx}(\ln x)\, dx\right]$$

$= x(\ln x)^2 - 2\left(\ln x \cdot x - \int x \cdot \frac{1}{x}\,dx\right)$

$= x(\ln x)^2 - 2\left[x\ln x - \int dx\right]$

$= x(\ln x)^2 - 2\left[x\ln x - x\right] + c$

$= x(\ln x)^2 - 2x\ln x + 2x + c$ Ans.

(xx) $\int \underset{I}{\ln|\tan x|} \cdot \underset{II}{\sec^2 x}\,dx$

$= \ln(\tan x)(\tan x) - \int \tan x \cdot \frac{d}{dx}(\ln \tan x)\,dx$

$= \tan x \cdot \ln[\tan x] - \int \tan x \cdot \frac{1}{\tan x} \cdot \sec^2 x\,dx$

$= \tan x \cdot \ln(\tan x) - \int \sec^2 x\,dx$

$= \tan x \ln(\tan x) - \tan x + c$ Ans.

(xxi) $\int \frac{x \sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Let $\sin^{-1}x = z$ $\qquad dz = \frac{dx}{\sqrt{1-x^2}}$

$\sin z = x$

$\Rightarrow \int \sin z \cdot z\,dz = \int \underset{I}{z}\,\underset{II}{\sin z}\,dz$

$= z(-\cos z) - \int(-\cos z)\frac{d(z)}{dz}\,dz$

$= -z\cos z + \int \cos z\,dz$

$= -z\cos z + \sin z + c$

Now $\sin z = x \Rightarrow \cos z = \sqrt{1-\sin^2 z}$

$\cos z = \sqrt{1-x^2}$

Thus $-\sin^{-1}x\sqrt{1-x^2} + x + c$

$= x - \sqrt{1-x^2}\sin^{-1}x + c$ Ans.

Q.2 Evaluate the following integrals:

(i) $\int \tan^4 x\,dx$

$= \int \tan^2 x \cdot \tan^2 x\,dx$

$= \int (\sec^2 x - 1)\tan^2 x\,dx$

$= \int (\sec^2 x \tan^2 x - \tan^2 x)\,dx$

$= \int \tan^2 x \cdot \sec^2 x\,dx - \int \tan^2 x\,dx$

$= \frac{1}{3}\tan^3 x - \int (\sec^2 x - 1)\,dx$

$= \frac{1}{3}\tan^3 x - \int \sec^2 x\,dx + \int dx$

$= \frac{1}{3}\tan^3 x - \tan x + x + c$ Ans.

$\because 1 + \tan^2\theta = \sec^2\theta$

(ii) $\int \sec^4 x\,dx$

$= \int \sec^2 x \cdot \sec^2 x\,dx$

$= \int (1 + \tan^2 x)\sec^2 x\,dx$

$= \int (\sec^2 x + \tan^2 x \cdot \sec^2 x)\,dx$

$= \int \sec^2 x\,dx + \int \tan^2 x\,\sec^2 x\,dx$

$= \tan x + \frac{1}{3}\tan^3 x + c$ Ans.

(iv) $\int \tan^3 x \sec x\,dx$

$= \int \tan^2 x \cdot (\tan x \sec x)\,dx$

$= \int (\sec^2 x - 1)\tan x \sec x\,dx$

$= \int \sec^2 x \cdot (\sec x \tan x)\,dx - \int \tan x \sec x\,dx$

$= \frac{1}{3}\sec^3 x - \sec x + c$ Ans.

(iii) $\int e^x \sin 2x \cos x \, dx$

$= \frac{1}{2}\int e^x [2\sin 2x \cos x]\, dx$

$\{\because 2\sin P \cos Q = \sin(P+Q) + \sin(P-Q)\}$

$= \frac{1}{2}\int e^x [\sin(2x+x) + \sin(2x-x)]\, dx$

$= \frac{1}{2}\int e^x (\sin 3x + \sin x)\, dx$

$= \frac{1}{2}\underbrace{\int e^x \sin 3x \, dx}_{I_1} + \frac{1}{2}\underbrace{\int e^x \sin x \, dx}_{I_2}$ —(1)

Let

$I_1 = \int \underset{I}{e^x} \underset{II}{\sin 3x}\, dx$

$I_1 = e^x\left(-\frac{\cos 3x}{3}\right) - \int\left(-\frac{\cos 3x}{3}\right)\frac{d}{dx}(e^x)\, dx$

$I_1 = -\frac{e^x \cos 3x}{3} + \frac{1}{3}\int \underset{I}{e^x} \underset{II}{\cos 3x}\, dx$

$I_1 = -\frac{e^x \cos 3x}{3} + \frac{1}{3}\left(e^x\left(\frac{\sin 3x}{3}\right) - \int \frac{\sin 3x}{3}\frac{de^x}{dx}\, dx\right)$

$I_1 = -\frac{e^x \cos 3x}{3} + \frac{1}{3}\left(\frac{e^x \sin 3x}{3} - \frac{1}{3}\int e^x \sin 3x \, dx\right)$

$I_1 = -\frac{e^x \cos 3x}{3} + \frac{e^x \sin 3x}{9} - \frac{1}{9} I_1$

$I_1 + \frac{1}{9} I_1 = -\frac{e^x \cos 3x}{3} + \frac{e^x \sin 3x}{9}$

$\left(1+\frac{1}{9}\right) I_1 = -\frac{e^x \cos 3x}{3} + \frac{e^x \sin 3x}{9}$

$\frac{10}{9} I_1 = -\frac{e^x \cos 3x}{3} + \frac{e^x \sin 3x}{9}$

$I_1 = \frac{9}{10}\left(-\frac{e^x \cos 3x}{3} + \frac{e^x \sin 3x}{9}\right)$

$I_1 = \frac{-9\, e^x \cos 3x}{10 \cdot 3} + \frac{9 e^x \sin 3x}{10 \cdot 9}$

$I_1 = \frac{-3 e^x \cos 3x}{10} + \frac{e^x \sin 3x}{10}$

Now $I_2 = \int \underset{I}{e^x} \underset{II}{\sin x}\, dx$

$I_2 = e^x(-\cos x) - \int (-\cos x)\frac{d}{dx}(e^x)\, dx$

$I_2 = -e^x \cos x + \int \underset{I}{e^x} \underset{II}{\cos x}\, dx$

$I_2 = -e^x \cos x + \left(e^x \sin x - \int \sin x \frac{d}{dx}(e^x)\, dx\right)$

$I_2 = -e^x \cos x + e^x \sin x - I_2$

$I_2 + I_2 = -e^x \cos x + e^x \sin x$

$2I_2 = -e^x \cos x + e^x \sin x$

$I_2 = \frac{1}{2}\left[-e^x \cos x + e^x \sin x\right]$

Putting I_1 and I_2 in Eq (1)

$= \frac{1}{2}\left(\frac{-3 e^x \cos 3x}{10} + \frac{e^x \sin 3x}{10}\right) + \frac{1}{2}\left[\frac{1}{2}\left(e^x \sin x - e^x \cos x\right)\right] + c$

$= \frac{-3 e^x \cos 3x}{20} + \frac{e^x \sin 3x}{20} + \frac{1}{4} e^x \sin x - \frac{1}{4} e^x \cos x + c$ Ans.

(v) $\int \underset{I}{x^3} \underset{II}{e^{5x}}\, dx$

$= x^3\left(\frac{e^{5x}}{5}\right) - \int \frac{e^{5x}}{5} \cdot \frac{d}{dx}(x^3)\, dx$

$= \frac{x^3 e^{5x}}{5} - \int \frac{e^{5x}}{5} \cdot 3x^2\, dx$

$= \frac{x^3 e^{5x}}{5} - \frac{3}{5}\int \underset{II}{e^{5x}} \cdot \underset{I}{x^2}\, dx$

$= \frac{x^3 e^{5x}}{5} - \frac{3}{5}\left[x^2 \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5} \cdot \frac{d}{dx}(x^2)\, dx\right]$

$= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{3}{5}\int \frac{e^{5x}}{5} \cdot 2x\, dx$

$= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6}{25}\int \underset{I}{x} \underset{II}{e^{5x}}\, dx$

$= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6}{25}\left[x \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5} \frac{dx}{dx}\, dx\right]$

$= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6x e^{5x}}{125} - \frac{6}{125}\int e^{5x}\, dx$

$= \frac{x^3 e^{5x}}{5} - \frac{3x^2 e^{5x}}{25} + \frac{6x e^{5x}}{125} - \frac{6 e^{5x}}{625} + c$ Ans.

(vi) $I=\int \underset{I}{e^{-x}} \underset{II}{\sin 2x}\, dx$

$I= e^{-x}\left(-\frac{\cos 2x}{2}\right)-\int\left(-\frac{\cos 2x}{2}\right)\frac{d}{dx}(e^{-x})dx$

$I=-\frac{e^{-x}\cos 2x}{2}-\int\left(-\frac{\cos 2x}{2}\right)(-e^{-x})\,dx$

$I=-\frac{e^{-x}\cos 2x}{2}-\frac{1}{2}\int \underset{I}{e^{-x}}\underset{II}{\cos 2x}\,dx$

$I=-\frac{e^{-x}\cos 2x}{2}-\frac{1}{2}\left(e^{-x}\left(\frac{\sin 2x}{2}\right)-\int\frac{\sin 2x}{2}\frac{d}{dx}(e^{-x})dx\right)$

$I=-\frac{e^{-x}\cos 2x}{2}-\frac{1}{2}\left(\frac{e^{-x}\sin 2x}{2}-\int\frac{\sin 2x}{2}(-e^{-x})dx\right)$

$I=-\frac{e^{-x}\cos 2x}{2}-\frac{e^{-x}\sin 2x}{4}-\frac{1}{4}\int e^{-x}\sin 2x\,dx$

$I=-\frac{e^{-x}\cos 2x}{2}-\frac{e^{-x}\sin 2x}{4}-\frac{1}{4}I$

$I+\frac{1}{4}I=-\frac{e^{-x}\cos 2x}{2}-\frac{e^{-x}\sin 2x}{4}$

$(1+\frac{1}{4})I=-\frac{e^{-x}\cos 2x}{2}-\frac{e^{-x}\sin 2x}{4}$

$\frac{5}{4}I=-\frac{e^{-x}\cos 2x}{2}-\frac{e^{-x}\sin 2x}{4}$

$I=\frac{4}{5}\left[-\frac{e^{-x}\cos 2x}{2}-\frac{e^{-x}\sin 2x}{4}\right]+c$ Ans.

(vii) $I=\int \underset{II}{e^{2x}}\underset{I}{\cos 3x}\,dx$

$I=\cos 3x\left(\frac{e^{2x}}{2}\right)-\int\frac{e^{2x}}{2}\frac{d}{dx}(\cos 3x)dx$

$I=\frac{e^{2x}\cos 3x}{2}-\int\frac{e^{2x}}{2}\cdot-\sin 3x\cdot 3\,dx$

$I=\frac{e^{2x}\cos 3x}{2}+\frac{3}{2}\int \underset{II}{e^{2x}}\underset{I}{\sin 3x}\,dx$

$I=\frac{e^{2x}\cos 3x}{2}+\frac{3}{2}\left(\sin 3x\left(\frac{e^{2x}}{2}\right)-\int\frac{e^{2x}}{2}\frac{d}{dx}(\sin 3x)dx\right)$

$I=\frac{e^{2x}\cos 3x}{2}+\frac{3}{2}\left(\frac{\sin 3x\, e^{2x}}{2}-\int\frac{e^{2x}}{2}\cos 3x\cdot 3dx\right)$

$I=\frac{e^{2x}\cos 3x}{2}+\frac{3}{4}e^{2x}\sin 3x-\frac{9}{4}I$

$I+\frac{9}{4}I=\frac{e^{2x}\cos 3x}{2}+\frac{3e^{2x}\sin 3x}{4}$

$\frac{13}{4}I=\frac{e^{2x}\cos 3x}{2}+\frac{3e^{2x}\sin 3x}{4}$

$I=\frac{4}{13}\left[\frac{e^{2x}\cos 3x}{2}+\frac{3e^{2x}\sin 3x}{4}\right]+c$

$I=\frac{2e^{2x}\cos 3x}{13}+\frac{3e^{2x}\sin 3x}{13}+c$ Ans.

(viii) $\int \csc^3 x\,dx$

$I=\int \underset{II}{\csc^2 x}\ \underset{I}{\csc x}\,dx$

$I=\csc x(-\cot x)-\int(-\cot x)\left(\frac{d}{dx}\csc x\right)dx$

$I=-\cot x\csc x+\int\cot x(-\csc x\cot x)dx$

$I=-\cot x\csc x-\int\cot^2 x\csc x\,dx$

$I=-\cot x\csc x-\int(\csc^2 x-1)\csc x\,dx$

$I=-\cot x\csc x-\int(\csc^3 x-\csc x)\,dx$

$I=-\cot x\csc x-\int\csc^3 x\,dx+\int\csc x\,dx$

$I=-\cot x.\csc x-I+\ln|\csc x-\cot x|$

$I+I=-\cot x\csc x+\ln(\csc x-\cot x)$

$2I=-\cot x.\csc x+\ln(\csc x-\cot x)$

$I=-\frac{\cot x\csc x}{2}+\frac{1}{2}\ln(\csc x-\cot x)+c$ Ans.

Q.3 $I=\int \underset{II}{e^{ax}}\underset{I}{\sin bx}\,dx$

$I=\sin bx\left(\frac{e^{ax}}{a}\right)-\int\frac{e^{ax}}{a}\frac{d}{dx}(\sin bx)dx$

$I=\frac{e^{ax}\sin bx}{a}-\int\frac{e^{ax}}{a}\cdot b\cos bx\,dx$

$I=\frac{e^{ax}\sin bx}{a}-\frac{b}{a}\int \underset{II}{e^{ax}}\underset{I}{\cos bx}\,dx$

$I=\frac{e^{ax}\sin bx}{a}-\frac{b}{a}\left(\cos bx\left(\frac{e^{ax}}{a}\right)-\int\frac{e^{ax}}{a}\frac{d}{dx}(\cos bx)dx\right)$

$I=\frac{e^{ax}\sin bx}{a}-\frac{b}{a}\left(\frac{\cos bx\, e^{ax}}{a}-\int\frac{e^{ax}}{a}(-\sin bx)b\cdot dx\right)$

$I=\frac{e^{ax}\sin bx}{a}-\frac{b e^{ax}\cos bx}{a^2}-\frac{b^2}{a^2}\int e^{ax}\sin bx\,dx$

$I=\frac{e^{ax}\sin bx}{a}-\frac{b e^{ax}\cos bx}{a^2}-\frac{b^2}{a^2}I$

$$I+\frac{b^2}{a^2}I = \frac{e^{ax}\sin bx}{a} - \frac{be^{ax}\cos bx}{a^2}$$

$$\left(\frac{a^2+b^2}{a^2}\right)I = \frac{e^{ax}\sin bx}{a} - \frac{be^{ax}\cos bx}{a^2}$$

$$I = \frac{a^2}{a^2+b^2}\left(\frac{e^{ax}\sin bx}{a} - \frac{b\,e^{ax}\cos bx}{a^2}\right)$$

$$I = \frac{ae^{ax}\sin bx}{a^2+b^2} - \frac{be^{ax}\cos bx}{a^2+b^2}$$

$$I = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)$$

Let $r\cos\theta = a$ $\quad$ $r\sin\theta = b$

$r^2\cos^2\theta = a^2$ $\quad$ $r^2\sin^2\theta = b^2$

$$r^2(\cos^2\theta+\sin^2\theta) = a^2+b^2$$

$$r^2 = a^2+b^2 \qquad \because \sin^2\theta+\cos^2\theta=1$$

$$r = \sqrt{a^2+b^2}$$

Also $\frac{r\sin\theta}{r\cos\theta} = \frac{b}{a} \Rightarrow \tan\theta = \frac{b}{a}$

$$\theta = \tan^{-1}(b/a)$$

$$I = \frac{e^{ax}}{a^2+b^2}\left(r\cos\theta\sin bx - r\sin\theta\cos bx\right)$$

$$I = \frac{re^{ax}}{a^2+b^2}\left(\sin bx\cos\theta - \cos bx\sin\theta\right)$$

$\because \sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta)$

$$I = \frac{re^{ax}}{a^2+b^2}\left[\sin(bx-\theta)\right]$$

Now putting r and θ, we have

$$I = \frac{\sqrt{a^2+b^2}\,e^{ax}}{a^2+b^2}\left[\sin(bx-\tan^{-1}(b/a))\right]+c$$

$$I = \frac{e^{ax}}{\sqrt{a^2+b^2}}\left[\sin(bx-\tan^{-1}(b/a))\right]+c$$

(Proved)

Q.4 Evaluate the following integrals:

(i) $\int\sqrt{a^2-x^2}\,dx$

Let $I = \int \underset{II}{1}\cdot\underset{I}{\sqrt{a^2-x^2}}\,dx$

$$I = x\sqrt{a^2-x^2} - \int x\frac{d}{dx}(a^2-x^2)^{\frac{1}{2}}dx$$

$$I = x\sqrt{a^2-x^2} - \int x\frac{-2x}{2\sqrt{a^2-x^2}}dx$$

$$I = x\sqrt{a^2-x^2} - \int\frac{-x^2}{\sqrt{a^2-x^2}}dx$$

$$I = x\sqrt{a^2-x^2} - \int\frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}dx$$

$$I = x\sqrt{a^2-x^2} - \int\frac{a^2-x^2}{\sqrt{a^2-x^2}} - \frac{a^2}{\sqrt{a^2-x^2}}dx$$

$$I = x\sqrt{a^2-x^2} - \int\frac{a^2-x^2}{\sqrt{a^2-x^2}}dx + \int\frac{a^2dx}{\sqrt{a^2-x^2}}$$

$$I = x\sqrt{a^2-x^2} - \int\sqrt{a^2-x^2}dx + a^2\int\frac{dx}{\sqrt{a^2-x^2}}$$

$$I = x\sqrt{a^2-x^2} - I + a^2\sin^{-1}\left(\frac{x}{a}\right)$$

$$I+I = x\sqrt{a^2-x^2} + a^2\sin^{-1}(x/a)$$

$$2I = x\sqrt{a^2-x^2} + a^2\sin^{-1}(x/a)$$

$$I = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c \quad \text{Ans.}$$

(ii) $\int\sqrt{x^2-a^2}\,dx$

Let $I = \int \underset{II}{1}\cdot\underset{I}{\sqrt{x^2-a^2}}$

$$I = \sqrt{x^2-a^2}.\,x - \int x\frac{d}{dx}(\sqrt{x^2-a^2})dx$$

$$I = x\sqrt{x^2-a^2} - \int x\frac{2x}{2\sqrt{x^2-a^2}}dx$$

$$I = x\sqrt{x^2-a^2} - \int\frac{x^2}{\sqrt{x^2-a^2}}$$

$$I = x\sqrt{x^2-a^2} - \int\frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}dx$$

$$I = x\sqrt{x^2-a^2} - \int\left(\frac{x^2-a^2}{\sqrt{x^2-a^2}} + \frac{a^2}{\sqrt{x^2-a^2}}\right)dx$$

$$I = x\sqrt{x^2-a^2} - \int\frac{x^2-a^2}{\sqrt{x^2-a^2}}dx - \int\frac{a^2}{\sqrt{x^2-a^2}}dx$$

$$I = x\sqrt{x^2-a^2} - \int\sqrt{x^2-a^2}\,dx - a^2\int\frac{dx}{\sqrt{x^2-a^2}}$$

$$I = x\sqrt{x^2-a^2} - I - a^2\cosh^{-1}\left(\frac{x}{a}\right)$$

$$I + I = x\sqrt{x^2-a^2} - a^2\cosh^{-1}(x/a)$$

$$2I = x\sqrt{x^2-a^2} - a^2\cosh^{-1}\left(\frac{x}{a}\right)$$

$$I = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + c$$

Ans.

Also

$$\cosh^{-1}\left(\frac{x}{a}\right) = \ln\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)$$

so

$$I = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\left[\frac{x+\sqrt{x^2-a^2}}{a}\right] + c$$

Ans.

(iii) $\int\sqrt{4-5x^2}\,dx$

$$= \int\sqrt{(2)^2-(\sqrt{5}x)^2}\,dx$$

Put $\sqrt{5}x = 2\sin\theta$

$\sqrt{5}dx = 2\cos\theta\,d\theta$

$dx = \frac{2}{\sqrt{5}}\cos\theta\,d\theta$

$$= \int\sqrt{(2)^2-(2\sin\theta)^2}\cdot\frac{2}{\sqrt{5}}\cos\theta\,d\theta$$

$$= \int\sqrt{4-4\sin^2\theta}\,\frac{2}{\sqrt{5}}\cos\theta\,d\theta$$

$$= \frac{2}{\sqrt{5}}\int\sqrt{4(1-\sin^2\theta)}\,\cos\theta\,d\theta$$

$$= \frac{2}{\sqrt{5}}\int 2\sqrt{\cos^2\theta}\cdot\cos\theta\,d\theta$$

$$= \frac{4}{\sqrt{5}}\int\cos\theta\cdot\cos\theta\,d\theta$$

$$= \frac{4}{\sqrt{5}}\int\cos^2\theta\,d\theta$$

$$= \frac{4}{\sqrt{5}}\int\frac{1+\cos 2\theta}{2}\,d\theta$$

$\because \cos 2\theta = 2\cos^2\theta - 1$, $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \frac{2}{\sqrt{5}}\int d\theta + \frac{2}{\sqrt{5}}\int\cos 2\theta\,d\theta$$

$$= \frac{2}{\sqrt{5}}\theta + \frac{2}{\sqrt{5}}\left(\frac{\sin 2\theta}{2}\right)$$

$$= \frac{2}{\sqrt{5}}\theta + \frac{1}{\sqrt{5}}\,2\sin\theta\cos\theta$$

Now $\sin\theta = \frac{\sqrt{5}}{2}x \Rightarrow \theta = \sin^{-1}\left(\frac{\sqrt{5}x}{2}\right)$

$$\cos\theta = \sqrt{1-\sin^2\theta}$$

$$\cos\theta = \sqrt{1-\left(\frac{\sqrt{5}}{2}x\right)^2} = \sqrt{1-\frac{5x^2}{4}}$$

$$\cos\theta = \sqrt{\frac{4-5x^2}{4}} = \frac{\sqrt{4-5x^2}}{2}$$

So

$$\int\sqrt{4-5x^2}\,dx = \frac{2}{\sqrt{5}}\sin^{-1}\left(\frac{\sqrt{5}x}{2}\right) + \frac{2}{\sqrt{5}}\frac{\sqrt{5}x}{2}\cdot\frac{\sqrt{4-5x^2}}{2}$$

$$\int\sqrt{4-5x^2}\,dx = \frac{2}{\sqrt{5}}\sin^{-1}\left(\frac{\sqrt{5}x}{2}\right) + \frac{x\sqrt{4-5x^2}}{2} + c$$

Ans.

(iv) $\int\sqrt{3-4x^2}\,dx$

$$= \int\sqrt{(\sqrt{3})^2-(2x)^2}\,dx$$

Put $2x = \sqrt{3}\sin\theta$ $\quad 2dx = \sqrt{3}\cos\theta\,d\theta$

$dx = \frac{\sqrt{3}}{2}\cos\theta\,d\theta$

$$= \int\sqrt{(\sqrt{3})^2-(\sqrt{3}\sin\theta)^2}\cdot\frac{\sqrt{3}}{2}\cos\theta\,d\theta$$

$$= \int\sqrt{3-3\sin^2\theta}\,d\theta\cdot\frac{\sqrt{3}}{2}\cos\theta$$

$$= \frac{\sqrt{3}}{2}\int\sqrt{3}\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

$$= \frac{3}{2}\int\sqrt{\cos^2\theta}\,\cos\theta\,d\theta$$

$$= \frac{3}{2}\int\cos^2\theta\,d\theta$$

$\because \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \frac{3}{2}\int\frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{3}{4}\int d\theta + \frac{3}{4}\int\cos 2\theta\,d\theta$$

$$= \frac{3}{4}\theta + \frac{3}{4}\left[\frac{\sin 2\theta}{2}\right]$$

$$= \frac{3}{4}\theta + \frac{3}{4}\cdot\frac{2\sin\theta\cos\theta}{2}$$

$$= \frac{3}{4}\theta + \frac{3}{4}\sin\theta\cos\theta$$

Now $\sin\theta = \frac{2x}{\sqrt{3}} \Rightarrow \theta = \sin^{-1}\left(\frac{2x}{\sqrt{3}}\right)$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{2x}{\sqrt{3}}\right)^2}$$

$$\cos\theta = \sqrt{1-\frac{4x^2}{3}} = \frac{\sqrt{3-4x^2}}{\sqrt{3}}$$

Thus $\int\sqrt{3-4x^2}\,dx = \frac{3}{4}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + \frac{3}{4}\cdot\frac{2x}{\sqrt{3}}\cdot\frac{\sqrt{3-4x^2}}{\sqrt{3}}$

$$\int\sqrt{3-4x^2}\,dx = \frac{3}{4}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + \frac{x\sqrt{3-4x^2}}{2} + c \quad \text{Ans.}$$

(v) $\int\sqrt{x^2+4}\,dx$

Let $I = \int \underset{II}{1}\cdot\underset{I}{\sqrt{x^2+4}}\,dx$

$$I = \sqrt{x^2+4}\cdot x - \int x\cdot\frac{d}{dx}\left(\sqrt{x^2+4}\right)dx$$

$$I = x\sqrt{x^2+4} - \int x\cdot\frac{2x}{2\sqrt{x^2+4}}\,dx$$

$$I = x\sqrt{x^2+4} - \int\frac{x^2}{\sqrt{x^2+4}}\,dx$$

$$I = x\sqrt{x^2+4} - \int\frac{x^2+4-4}{\sqrt{x^2+4}}\,dx$$

$$I = x\sqrt{x^2+4} - \int\frac{x^2+4}{\sqrt{x^2+4}}\,dx + \int\frac{4}{\sqrt{x^2+4}}\,dx$$

$$I = x\sqrt{x^2+4} - \int\sqrt{x^2+4}\,dx + 4\int\frac{dx}{\sqrt{x^2+4}}$$

$$I = x\sqrt{x^2+4} - I + 4\sinh^{-1}\left(\frac{x}{2}\right)$$

$$I + I = x\sqrt{x^2+4} + 4\sinh^{-1}\left(\frac{x}{2}\right)$$

$$2I = x\sqrt{x^2+4} + 4\sinh^{-1}\left(\frac{x}{2}\right)$$

$$I = \frac{x\sqrt{x^2+4}}{2} + \frac{4\sinh^{-1}\left(\frac{x}{2}\right)}{2} + c$$

$$I = \frac{x\sqrt{x^2+4}}{2} + 2\sinh^{-1}\left(\frac{x}{2}\right) + c$$

Ans.

(vi) $\int \underset{I}{x^2}\,\underset{II}{e^{ax}}\,dx$

$$= x^2\left(\frac{e^{ax}}{a}\right) - \int\frac{e^{ax}}{a}\frac{d}{dx}(x^2)\,dx$$

$$= \frac{e^{ax}x^2}{a} - \int\frac{e^{ax}}{a}\,2x\,dx$$

$$= \frac{e^{ax}x^2}{a} - \frac{2}{a}\int \underset{I}{x}\,\underset{II}{e^{ax}}\,dx$$

$$= \frac{e^{ax}x^2}{a} - \frac{2}{a}\left[x\left(\frac{e^{ax}}{a}\right) - \int\frac{e^{ax}}{a}\frac{d(x)}{dx}\,dx\right]$$

$$= \frac{e^{ax}x^2}{a} - \frac{2xe^{ax}}{a^2} + \frac{2}{a^2}\int e^{ax}\,dx$$

$$= \frac{x^2e^{ax}}{a} - \frac{2xe^{ax}}{a^2} + \frac{2}{a^2}\left(\frac{e^{ax}}{a}\right) + c$$

$$= \frac{x^2e^{ax}}{a} - \frac{2xe^{ax}}{a^2} + \frac{2e^{ax}}{a^3} + c \quad \text{Ans.}$$

The Useful integral:-

If The function $f(x)$ is then

$$\int e^x\left[f(x)+f'(x)\right]dx = e^x f(x).$$

$$\int e^{ax}\left[af(x)+f'(x)\right]dx = e^{ax}f(x)$$

Q.5 Evaluate the following integrals:

(i) $\int e^x\left[\ln x + \frac{1}{x}\right]dx$

$$\because \int e^x\left[f(x)+f'(x)\right]dx = e^x f(x)$$

$$\Rightarrow e^x(\ln x) + c \quad \text{Ans.}$$

(ii) $\int e^x(\sin x + \cos x)\,dx$

$$\because \int e^x\left[f(x)+f'(x)\right]dx = e^x f(x)$$

$$\Rightarrow \int e^x(\sin x+\cos x)\,dx = e^x\sin x + c \quad \text{Ans.}$$

(iii) $\int e^{ax}\left[a\sec^{-1}x + \frac{1}{x\sqrt{x^2-1}}\right]dx$

$\because \int e^{ax}[af(x)+f'(x)]dx = e^{ax}f(x)$

$\Rightarrow \int e^{ax}\left[a\sec^{-1}x + \frac{1}{x\sqrt{x^2-1}}\right]dx = e^{ax}\sec^{-1}x + c$ Ans

(iv) $\int e^{3x}\left[\frac{3\sin x - \cos x}{\sin^2 x}\right]dx$

$= \int e^{3x}\left[\frac{3\sin x}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\right]dx$

$= \int e^{3x}\left[\frac{3}{\sin x} - \frac{\cos x}{\sin x}\cdot\frac{1}{\sin x}\right]dx$

$= \int e^{3x}\left[3\operatorname{cosec} x - \cot x\cdot\operatorname{cosec} x\right]dx$

$= \int e^{3x}\left[3\operatorname{cosec} x + (-\cot x\operatorname{cosec} x)\right]dx$

$= e^{3x}\cdot\operatorname{cosec} x + c$ Ans

(v) $\int e^{2x}\left[-\sin x + 2\cos x\right]dx$

$= \int e^{2x}\left[2\cos x + (-\sin x)\right]dx$

$= e^{2x}\cos x + c$ Ans

(vi) $\int \frac{xe^x}{(1+x)^2}dx$

$= \int e^x\left[\frac{x}{(1+x)^2}\right]dx$

$= \int e^x\left(\frac{x+1-1}{(x+1)^2}\right)dx$

$= \int e^x\left[\frac{(x+1)}{(x+1)^2} + \frac{-1}{(1+x)^2}\right]dx$

$= \int e^x\left[\frac{1}{(x+1)} + \frac{-1}{(x+1)^2}\right]dx$

$= e^x\cdot\frac{1}{(x+1)} + c$

$= \frac{e^x}{(1+x)} + c$ Ans

(vii) $\int e^{-x}(\cos x - \sin x)dx$

$= \int e^{-x}\left[-\sin x + \cos x\right]dx$

$= \int e^{-x}\left[(-1)\sin x + \cos x\right]dx$

$= e^{-x}\sin x + c$ Ans

(viii) $\int \frac{e^{m\tan^{-1}x}}{(1+x^2)}dx$

Let $\tan^{-1}x = z$

$\frac{1}{1+x^2}dx = dz \Rightarrow dx = dz(1+x^2)$

$= \int \frac{e^{mz}}{\cancel{(1+x^2)}}\cdot dz\,\cancel{(1+x^2)}$

$= \int e^{mz}dz$

$= \frac{e^{mz}}{m} + c$ Now $z = \tan^{-1}x$ back

$= \frac{e^{m\tan^{-1}x}}{m} + c$ Ans

(ix) $\int \frac{2(x-1)}{1+\cos 2x}dx$

$\because \cos 2x = 2\cos^2 x - 1$
$\Rightarrow 1+\cos 2x = 2\cos^2 x$

$= \int \frac{2x-2}{2\cos^2 x}dx$

$= \int \frac{\cancel{2}x\,dx}{\cancel{2}\cos^2 x} - \int \frac{\cancel{2}}{\cancel{2}\cos^2 x}dx$

$= \int \underset{I}{x}\,\underset{II}{\sec^2 x}\,dx - \int \sec^2 x\,dx$

$= x\tan x - \int \tan x\frac{d(x)}{dx}dx - \tan x$

$= x\tan x - \int \tan x\,dx - \tan x$

$= x\tan x - \ln|\sec x| - \tan x + c$ Ans

or $= x\tan x + \ln|\sec x|^{-1} - \tan x + c$

$= x\tan x + \ln\left|\frac{1}{\sec x}\right| - \tan x + c$

$= x\tan x + \ln|\cos x| - \tan x + c$ Ans

$= (x-1)\tan x + \ln|\cos x| + c$

(x) $\int \frac{2x}{1-\cos x}\,dx$

$= \int \frac{2x\,dx}{2\sin^2 x/2}$ $\quad \because \cos x = 1-2\sin^2\frac{x}{2}$, $\; 2\sin^2\frac{x}{2} = 1-\cos x$

$= \int \underset{I}{x}\cdot \underset{II}{\operatorname{cosec}^2\frac{x}{2}}\,dx$

$= x\cdot\left(\frac{-\cot x/2}{1/2}\right) - \int -\frac{\cot\frac{x}{2}}{1/2}\cdot\frac{d(x)}{dx}\cdot dx$

$= -x\cdot 2\cot\frac{x}{2} + 2\int \cot\frac{x}{2}\,dx$

$= -2x\cot x/2 + 2\int \frac{\cos x/2}{\sin x/2}\,dx$

$= -2x\cot\frac{x}{2} + 4\int \frac{\frac{1}{2}\cos\frac{x}{2}}{\sin x/2}\,dx$

$= -2x\cot\frac{x}{2} + 4\ln|\sin\frac{x}{2}| + c$ Ans.

(xi) $\int \frac{2x}{1-\sin x}\,dx$

$= \int \frac{2x}{1-\sin x}\times\frac{1+\sin x}{1+\sin x}\,dx$

$= \int \frac{2x(1+\sin x)}{1-\sin^2 x}\,dx$

$= \int \frac{2x+2x\sin x}{\cos^2 x}\,dx$

$= \int \frac{2x}{\cos^2 x}\,dx + \int \frac{2x\sin x}{\cos^2 x}\,dx$

$= \int 2\underset{I}{x}\,\underset{II}{\sec^2 x}\,dx + \int 2\underset{I}{x}\,\underset{II}{(\tan x\cdot\sec x)}\,dx$

$= 2\left[x\tan x - \int \tan x\cdot\frac{d(x)}{dx}dx\right] + 2\left(x\sec x - \int \sec x\frac{d(x)}{dx}dx\right)$

$= 2(x\tan x - \int \tan x\,dx) + 2(x\sec x - \int \sec x\,dx)$

$= 2[x\tan x - \ln|\sec x|] + 2[x\sec x - \ln|\tan x+\sec x|]$

$= 2x\tan x - 2\ln|\sec x| + 2x\sec x - 2\ln|\tan x+\sec x| + c$

$= 2x(\tan x+\sec x) - 2\ln|(\sec x)(\tan x+\sec x)| + c$ Ans.

(xii) $\int \frac{e^x(1+x)}{(2+x)^2}\,dx$

$= \int e^x\left[\frac{1+x+1-1}{(2+x)^2}\right]dx$

$= \int e^x\left[\frac{2+x}{(2+x)^2} + \frac{-1}{(2+x)^2}\right]dx$

$= \int e^x\left[\frac{1}{(2+x)} + \frac{-1}{(2+x)^2}\right]dx$

$= e^x\frac{1}{(2+x)} + c$ Ans.

(xiii) $\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

$= \int e^x\left[\frac{1-2\sin x/2\cos x/2}{2\sin^2 x/2}\right]dx$

$= \int e^x\left(\frac{1}{2\sin^2\frac{x}{2}} - \frac{\cancel{2}\sin x/2\cos x/2}{\cancel{2}\sin^2 x/2}\right)dx$

$\int e^x\left[\frac{1}{2}\operatorname{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right]dx$

$\int e^x\left[\left(-\cot\frac{x}{2}\right) + \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right]dx$

$= -e^x\cot\frac{x}{2} + c$ Ans.

(xiv) $\int \frac{x+\sin x}{1+\cos x}\,dx$

$= \int \frac{x+2\sin x/2\cos x/2}{2\cos^2 x/2}\,dx$

$= \int\left(\frac{x}{2\cos^2 x/2} + \frac{\cancel{2}\sin x/2\cos x/2}{\cancel{2}\cos^2 x/2}\right)dx$

$= \int \frac{x}{2}\sec^2\frac{x}{2}\,dx + \int \tan\frac{x}{2}\,dx$

$= \int \underset{I}{x}\cdot\underset{II}{\frac{1}{2}\sec^2\frac{x}{2}}\,dx + \int \tan\frac{x}{2}\,dx$

$= x\tan\frac{x}{2} - \cancel{\int \tan\frac{x}{2}\cdot 1dx} + \cancel{\int \tan\frac{x}{2}dx}$

$= x\tan\frac{x}{2} + c$ Ans.

(xv)

$$\int \frac{x^2}{(x\sin x+\cos x)^2} \qquad \text{Let } u=(\cos x+x\sin x)$$

$$=\int \frac{x\cos x\cdot \frac{x}{\cos x}\,dx}{(x\sin x+\cos x)^2} \qquad \frac{du}{dx}=-\sin x+x\cos x+1\cdot\sin x$$

$$\frac{du}{dx}=x\cos x$$

$$=\int\left[(x\sin x+\cos x)^{-2}\cdot x\cos x\right]\frac{x}{\cos x}\,dx \qquad \left\{\because \int f'(x)\,dx=f(x)\right\}$$

$$=\int\left[\underset{\text{II}}{(x\sin x+\cos x)^{-2}x\cos x}\right]\underset{\text{I}}{(x\sec x)}\,dx$$

$$=(x\sec x)\cdot\frac{(x\sin x+\cos x)^{-1}}{-1}-\int\frac{(x\sin x+\cos x)^{-1}}{-1}\frac{d}{dx}(x\sec x)\,dx$$

$$=\frac{-x\sec x}{(x\sin x+\cos x)}+\int\frac{\sec x+x\sec x\tan x}{(x\sin x+\cos x)}\,dx$$

$$=\frac{-x\sec x}{(x\sin x+\cos x)}+\int\frac{\sec x(1+x\tan x)}{(x\sin x+\cos x)}\,dx$$

$$=\frac{-x\sec x}{(x\sin x+\cos x)}+\int\frac{\sec x\left(1+x\frac{\sin x}{\cos x}\right)}{(x\sin x+\cos x)}\,dx$$

$$=\frac{-x\sec x}{(x\sin x+\cos x)}+\int\frac{\sec x}{\cos x}\frac{\cancel{(\cos x+x\sin x)}}{\cancel{(x\sin x+\cos x)}}\,dx$$

$$=\frac{-x\sec x}{(x\sin x+\cos x)}+\int\sec^2 x\,dx$$

$$=\frac{-x\sec x}{(x\sin x+\cos x)}+\tan x+c \quad \text{Ans.}$$

Integration By Partial Fraction:-

"Using the idea of partial fraction we can evaluate some complicated and rational function by inverting according to the rules of integration. This process of evaluating the integrals is called integration by partial Fraction."