Integration:-

If $f'(x)$ is a derivative of $f(x)$ then the function $f(x)$ is called its anti derivate and is written as

$$\int f'(x)\,dx = f(x) + c \leftarrow \text{(Also called indefinate integration)}$$

where "$\int$" is integral sign "$f(x)$" is integrand and "dx" is called differential of x while x is called variable of integration and "c" is abitrary constant and also called constant of integration.

"Some Useful Formulas"

1) $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$

2) $\int \sin x\,dx = -\cos x + c$

3) $\int \cos x\,dx = \sin x + c$

4) $\int \sec^2 x\,dx = \tan x + c$

5) $\int \csc^2 x\,dx = -\cot x + c$

6) $\int \sec x \cdot \tan x\,dx = \sec x + c$

7) $\int \csc x \cdot \cot x\,dx = -\csc x + c$

8) $\int e^x\,dx = e^x + c$

9) $\int \frac{1}{x}\,dx = \ln x + c$

10) $\int a^x\,dx = \dfrac{a^x}{\ln a} + c$

11) $\int \tan x\,dx = \ln|\sec x| + c$

12) $\int \cot x\,dx = \ln|\sin x|$

13) $\int \sec x\,dx = \ln|\sec x + \tan x| + c$

14) $\int \csc x\,dx = \ln|\csc x - \cot x| + c$

15) $\int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$

16) $\int \dfrac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + c$

17) $\int \dfrac{dx}{1+x^2} = \tan^{-1} x + c$

18) $\int \dfrac{-1}{1+x^2}\,dx = \cot^{-1} x + c$

19) $\int \dfrac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + c$

20) $\int \dfrac{-1}{x\sqrt{x^2-1}}\,dx = \csc^{-1} x + c$

21) $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a} \ln\left|\dfrac{x-a}{x+a}\right| + c$

22) $\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a} \ln\left|\dfrac{a+x}{a-x}\right| + c$

23) $\int \dfrac{dx}{\sqrt{x^2-1}} = \ln(x+\sqrt{x^2-1})$ or $\cosh^{-1} x + c$

24) $\int \dfrac{dx}{\sqrt{1+x^2}} = \ln(x+\sqrt{x^2+1})$ or $\sinh^{-1} x + c$

25) First rule

$$\int [f(x)]^n \cdot f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1} + c$$

26) 2nd Rule

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

27) $\int 1\,dx = x + c$

28) $\int a\,dx = a \cdot x + c$

Exercise: 3.2

Q.1 Evaluate the following indefinate integrals:

(i) $\int (3x^2 - 2x + 1)\, dx$

$= 3\int x^2 dx - 2\int x\, dx + \int 1 dx$

$= 3\left(\frac{x^{2+1}}{2+1}\right) - 2\left(\frac{x^{1+1}}{1+1}\right) + x + c$

$= 3\frac{x^3}{3} - 2\frac{x^2}{2} + x + c$

$= x^3 - x^2 + x + c$ Ans:

(ii) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx \quad (x>0)$

$= \int \sqrt{x}\, dx + \int \frac{1}{\sqrt{x}}\, dx$

$= \int x^{1/2} dx + \int x^{-1/2} dx$

$= \frac{x^{1/2+1}}{1/2+1} + \frac{x^{-1/2+1}}{-1/2+1} + c$

$= \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + c$

$= \frac{2}{3}x^{3/2} + 2x^{1/2} + c$ Ans:

(iii) $\int x(\sqrt{x} + 1)\, dx \quad (x>0)$

$= \int x\sqrt{x}\, dx + \int x\, dx$

$= \int x^{3/2} dx + \int x\, dx$

$= \frac{x^{3/2+1}}{3/2+1} + \frac{x^{1+1}}{1+1} + c$

$= \frac{2}{5}x^{5/2} + \frac{1}{2}x^2 + c$ Ans:

(iv) $\int (2x+3)^{1/2} dx$

$= \frac{1}{2}\int (2x+3)^{1/2} \cdot 2\, dx$

$\because \int [f(x)]^n f'(x)\, dx = \frac{(f(x))^{n+1}}{n+1} + c$

$= \frac{1}{2}\frac{(2x+3)^{1/2+1}}{1/2+1} + c$

$= \frac{1}{2}\cdot\frac{(2x+3)^{3/2}}{3/2} + c$

$= \frac{1}{3}(2x+3)^{3/2} + c$ Ans:

(v) $\int (\sqrt{x}+1)^2 dx \quad (x>0)$

$= \int [(\sqrt{x})^2 + (1)^2 + 2(\sqrt{x})]\, dx$

$= \int (x + 1 + 2\sqrt{x})\, dx$

$= \int x\, dx + \int 1\cdot dx + 2\int x^{1/2} dx$

$= \frac{x^{1+1}}{1+1} + x + 2\cdot\frac{x^{1/2+1}}{1/2+1} + c$

$= \frac{1}{2}x^2 + x + 2\cdot\frac{2}{3}x^{3/2} + c$

$= \frac{1}{2}x^2 + \frac{4}{3}x^{3/2} + x + c$ Ans:

(vi) $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx \quad (x>0)$

$= \int \left[(\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2(\sqrt{x})\left(\frac{1}{\sqrt{x}}\right)\right] dx$

$= \int \left(x + \frac{1}{x} - 2\right) dx$

$= \int x\, dx + \int \frac{1}{x} dx - 2\int 1 dx$

$= \frac{x^{1+1}}{1+1} + \ln x - 2x + c$

$= \frac{1}{2}x^2 - 2x + \ln x + c$ Ans:

(vii) $\int \frac{3x+2}{\sqrt{x}}\, dx \quad (x>0)$

$= \int \frac{3x}{\sqrt{x}}\, dx + \int \frac{2}{\sqrt{x}}\, dx$

$= 3\int \frac{x}{\sqrt{x}}\, dx + 2\int x^{-1/2} dx$

$= 3\int x^{1/2} dx + 2\int x^{-1/2} dx \quad \left(\because \frac{x}{\sqrt{x}} = \sqrt{x}\right)$

$= 3\cdot\frac{x^{1/2+1}}{1/2+1} + 2\cdot\frac{x^{-1/2+1}}{-1/2+1} + c$

$= 3\cdot\frac{2}{3}x^{3/2} + 2\cdot\frac{2}{1}x^{1/2} + c$

$2x^{3/2} + 4x^{1/2} + c$ Ans:

(viii) $\int \frac{\sqrt{y}(y+1)}{y} dy \qquad (y>0)$

$= \int \left(\frac{y\sqrt{y} + \sqrt{y}}{y}\right) dy$

$= \int \frac{y\sqrt{y}}{y} dy + \int \frac{\sqrt{y}}{y} dy$

$= \int \sqrt{y}\, dy + \int \frac{1}{\sqrt{y}} dy \qquad \left(\because \frac{\sqrt{y}}{y} = \frac{1}{\sqrt{y}}\right)$

$= \int y^{1/2} dy + \int y^{-1/2} dy$

$= \frac{y^{1/2+1}}{1/2+1} + \frac{y^{-1/2+1}}{-1/2+1} + c$

$= \frac{2}{3} y^{3/2} + \frac{2}{1} y^{1/2} + c$

$= \frac{2}{3} y^{3/2} + 2y^{1/2} + c$ Ans.

(ix) $\int \frac{(\sqrt{\theta}-1)^2}{\sqrt{\theta}} d\theta \qquad (\theta>0)$

$= \int \frac{\theta+1-2\sqrt{\theta}}{\sqrt{\theta}} d\theta$

$= \int \frac{\theta}{\sqrt{\theta}} d\theta + \int \frac{1}{\sqrt{\theta}} d\theta - 2\int \frac{\sqrt{\theta}}{\sqrt{\theta}} d\theta$

$= \int \sqrt{\theta}\, d\theta + \int \theta^{-1/2} d\theta - 2\int d\theta$

$= \frac{\theta^{1/2+1}}{1/2+1} + \frac{\theta^{-1/2+1}}{-1/2+1} - 2\theta + c$

$= \frac{2}{3}\theta^{3/2} + 2\theta^{1/2} - 2\theta + c$ Ans.

(x). $\int \frac{(1-\sqrt{x})^2}{\sqrt{x}} dx \qquad (x>0)$

$= \int \frac{1+x-2\sqrt{x}}{\sqrt{x}} dx$

$= \int \frac{1}{\sqrt{x}} dx + \int \frac{x}{\sqrt{x}} dx - 2\int \frac{\sqrt{x}}{\sqrt{x}} dx$

$= \int x^{-1/2} dx + \int x^{1/2} dx - 2\int dx$

$= \frac{x^{-1/2+1}}{-1/2+1} + \frac{x^{1/2+1}}{1/2+1} - 2x + c$

$= 2x^{1/2} + \frac{2}{3} x^{3/2} - 2x + c$ Ans

(xi) $\int \frac{e^{2x} + e^{x}}{e^{x}} dx$

$= \int \frac{e^{2x}}{e^{x}} dx + \int \frac{e^{x}}{e^{x}} dx$

$= \int \frac{e^{x} \cdot e^{x}}{e^{x}} dx + \int 1 \cdot dx$

$= \int e^{x} dx + \int 1\, dx$

$= e^{x} + x + c$ Ans.

Q.2 Evaluate

(i) $\int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}} \qquad \begin{pmatrix} x+a>0 \\ x+b>0 \end{pmatrix}$

$= \int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{\sqrt{x+a} - \sqrt{x+b}}{\sqrt{x+a} - \sqrt{x+b}}$

$= \int \frac{(\sqrt{x+a} - \sqrt{x+b})\, dx}{(\sqrt{x+a})^2 - (\sqrt{x+b})^2}$

$= \int \frac{(\sqrt{x+a} - \sqrt{x+b})\, dx}{x+a-x-b}$

$= \int \frac{\sqrt{x+a} - \sqrt{x+b}}{a-b} dx$

$= \frac{1}{a-b} \int \sqrt{x+a}\, dx - \frac{1}{a-b} \int \sqrt{x+b}\, dx$

$= \frac{1}{a-b} \left[\frac{(x+a)^{1/2+1}}{1/2+1} - \frac{(x+b)^{1/2+1}}{1/2+1} \right]$

$= \frac{1}{a-b} \left(\frac{2}{3}(x+a)^{3/2} - \frac{2}{3}(x+b)^{3/2} \right)$

$= \frac{2}{3(a-b)} \left[(x+a)^{3/2} - (x+b)^{3/2} \right]$ Ans.

(ii) $\int \frac{1-x^2}{1+x^2} dx$ By add & Sub. 1

$= \int \frac{1-x^2-1+1}{1+x^2}$

$= \int \frac{2-(x^2+1)}{1+x^2} dx \Rightarrow \int \frac{2}{1+x^2} dx - \int \frac{1+x^2}{1+x^2} dx$

$= 2\int \frac{dx}{1+x^2} - \int dx = 2\tan^{-1}x - x + c$ Ans.

(iii) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$

$= \int \frac{dx}{\sqrt{x+a}+\sqrt{x}} \times \frac{\sqrt{x+a}-\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}$

$= \int \frac{(\sqrt{x+a}-\sqrt{x})dx}{(\sqrt{x+a})^2-(\sqrt{x})^2}$

$= \int \frac{(\sqrt{x+a}-\sqrt{x})^2 dx}{x+a-x} \Rightarrow \int \frac{(\sqrt{x+a}-\sqrt{x})dx}{a}$

$= \frac{1}{a}\int \sqrt{x+a}\,dx - \frac{1}{a}\int \sqrt{x}\,dx$

$\frac{1}{a}\left[\frac{2}{3}(x+a)^{3/2} - \frac{2}{3}x^{3/2}\right] + c$

$= \frac{2}{3a}\left[(x+a)^{3/2} - x^{3/2}\right] + c$ Ans

(iv) $\int (a-2x)^{3/2} dx$

$= \frac{-1}{2}\int (a-2x)^{3/2} \cdot (-2)\, dx$

$= \frac{-1}{2}\int (a-2x)^{3/2}(-2)\, dx$

$= \frac{-1}{2}\left[\frac{(a-2x)^{3/2+1}}{3/2+1}\right] + c$

$= \frac{-1}{2}\left[\frac{2}{5}(a-2x)^{5/2}\right] + c$

$= \frac{-1}{5}(a-2x)^{5/2} + c$ Ans.

(v) $\int \frac{(1+e^x)^3}{e^x} dx$

$= \int \frac{(1+e^{3x}+3e^x+3e^{2x})\,dx}{e^x}$

$\because (a+b)^3 = a^3+b^3+3a^2b+3ab^2$

$= \int \frac{1}{e^x}dx + \int \frac{e^{3x}}{e^x}dx + 3\int \frac{e^x}{e^x}dx + 3\int \frac{e^{2x}}{e^x}dx$

$= \int e^{-x}dx + \int e^{2x}dx + 3\int dx + 3\int e^x dx$

$= -1\int e^{-x}(-1)dx + \frac{1}{2}\int e^{2x}\cdot 2dx + 3\int dx + 3\int e^x dx$

$= -e^{-x} + \frac{1}{2}e^{2x} + 3x + 3e^x + c$ Ans.

(vi) $\int \sin[(a+b)x]\, dx$

$= \frac{1}{a+b}\int \sin[(a+b)x]\cdot(a+b)\,dx$

$= \frac{1}{a+b}\left[-\cos(a+b)x\right] + [c]$

$= \frac{-\cos(a+b)x}{a+b} + c$ Ans.

(vii) $\int \sqrt{1-\cos 2x}\, dx$

$\because \cos 2x = 1 - 2\sin^2 x$

$= \int \sqrt{1-(1-2\sin^2 x)}\, dx$

$= \int \sqrt{1-1+2\sin^2 x}\, dx$

$= \int \sqrt{2\sin^2 x}\, dx$

$= \sqrt{2}\int \sin x\, dx$

$= \sqrt{2}\left[-\cos x\right] + c$

$= -\sqrt{2}\cos x + c$ Ans.

(viii) $\int \ln(x)\cdot \frac{1}{x}\, dx$

$= \int [\ln(x)]\cdot(1/x)\, dx$

$\because \int [f(x)]^n \cdot f'(x)\, dx = \frac{(f(x))^{n+1}}{n+1} + c$

$= \frac{[\ln(x)]^{1+1}}{1+1} + c$

$= \frac{1}{2}(\ln x)^2 + c$ Ans.

(ix) $\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}\, dx$

$\because \cos 2x = 1-2\sin^2 x$, $\sin^2 x = \frac{1-\cos 2x}{2}$

$= \frac{1}{2}\int 1\, dx - \frac{1}{2}\int \cos 2x\, dx$

$= \frac{1}{2}x - \frac{1}{2}\cdot\frac{\sin 2x}{2}$

$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c$ Ans.

(x) $\int \frac{1}{1+\cos x}\,dx$

$= \int \frac{1}{1+\cos x} \times \frac{1-\cos x}{1-\cos x}\,dx$

$= \int \frac{1-\cos x}{1-\cos^2 x}\,dx$

$= \int \frac{1-\cos x}{\sin^2 x}\,dx \Rightarrow \int\left(\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\right)dx$

$= \int \operatorname{cosec}^2 x\,dx - \int \cot x \cdot \operatorname{cosec} x\,dx$

$= -\cot x - (-\operatorname{cosec} x) + c$

$= = \cot x + \operatorname{cosec} x + c$ Ans.

(xi) $\int \frac{ax+b}{ax^2+2bx+c}\,dx$

$= \frac{1}{2}\int \frac{2(ax+b)}{ax^2+2bx+c}\,dx$

$= \frac{1}{2}\int \frac{2ax+2b}{ax^2+2bx+c}\,dx$

$= \frac{1}{2} \ln|ax^2+2bx+c| + c$

$\because \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

(xii) $\int \cos 3x \sin 2x\,dx$

$\because 2\cos P \sin Q = \sin(P+Q) - \sin(P-Q)$

$\Rightarrow \frac{1}{2}\int (2\cos 3x \sin 2x)\,dx$

$= \frac{1}{2}\int [\sin(3x+2x) - \sin(3x-2x)]\,dx$

$= \frac{1}{2}\int (\sin 5x - \sin x)\,dx$

$= \frac{1}{2}\int \sin 5x\,dx - \frac{1}{2}\int \sin x\,dx$

$= \frac{1}{2}\left(\frac{-\cos 5x}{5}\right) - \frac{1}{2}(-\cos x) + c$

$= -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + c$ Ans.

(xiii) $\int \frac{\cos 2x - 1}{1+\cos 2x}\,dx$

$= \int \frac{(1-2\sin^2 x) - 1}{1+(2\cos^2 x - 1)}\,dx$ $\quad \because \cos 2x = 1-2\sin^2 x = 2\cos^2 x - 1$

$= \int \frac{1-2\sin^2 x - 1}{1+2\cos^2 x - 1}\,dx$

$= \int \frac{-2\sin^2 x}{2\cos^2 x}\,dx$

$= -\int \tan^2 x\,dx$ $\quad \because 1+\tan^2 x = \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$

$= -\int (\sec^2 x - 1)\,dx$

$= -\int \sec^2 x\,dx + \int 1\,dx$

$= -\tan x + x + c$ Ans.

(xiv) $\int \tan^2 x\,dx$

$= \int (\sec^2 x - 1)\,dx$ $\quad \because 1+\tan^2 x = \sec^2 x$, $\tan^2 x = \sec^2 x - 1$

$= \int \sec^2 x\,dx - \int 1\,dx$

$= \tan x - x + c$ Ans.

Integration by Substitutions:-

To evaluate some indefinate integrals, we suppose some term to any variable to evaluate it by simple method reducing complicated and difficult methods and then substitute the values of variable and its differential in the given function.