Tahir Mahmood
M.Sc (Math)
Mob No: 03[illegible]-6510779

The Right Hand Limit:-

Let $f(x)$ be any function then

$$\lim_{x\to a^+} f(x) = L$$

is called right hand limit where x approaches to "a" for $x>a$

The Left Hand Limit:-

Let $f(x)$ be any function then

$$\lim_{x\to a^-} f(x) = L$$

is called left hand limit where x approaches to "a" for $x<a$

* Both Right and Left hand limits are called one sided limits.
* Both Right and Left hand limits are calculated by same criteria as usual limit (Two sided limit).

Continuous Function:-

A function f is said to be continuous at $x=a$ if

(i) $f(a)$ is a definite number.

(ii) $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$

(iii) $\lim_{x\to a} f(x) = f(a)$

* If $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$ then $\lim_{x\to a} f(x)$ exist and is L.

Discontinuous Function:-

"Any function which does not satisfy the conditions of continuous function is called discontinuous function."

Exercise 1.4

Q.1 Find right and left hand limits and find limit at $x=c$.

(i) $f(x) = 2x^2+x-5 \qquad c=1$

Right Hand Limit

$$\lim_{x\to c^+} f(x)$$
$$=\lim_{x\to 1^+}(2x^2+x-5)$$
$$=2(1)^2+(1)-5$$
$$=3-5=-2$$
$$\lim_{x\to 1^+} f(x) = -2$$

Left Hand Limit.

$$\lim_{x\to c^-} f(x)$$
$$\lim_{x\to 1^-}(2x^2+x-5)$$
$$=2(1)^2+(1)-5$$
$$=3-5=-2$$
$$\lim_{x\to 1^-} f(x) = -2$$

$\because \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)$

Hence limit exists and is

$$\lim_{x\to 1} f(x) = -2$$

(ii) $f(x) = \dfrac{x^2-9}{x-3} \qquad c=-3$

Right Hand Limit

$$\lim_{x\to -3^+} f(x)$$
$$=\lim_{x\to -3^+}\frac{x^2-9}{x-3}$$
$$=\lim_{x\to -3^+}\frac{(\cancel{x-3})(x+3)}{(\cancel{x-3})}$$
$$=\lim_{x\to -3^+}(x+3)$$
$$=(-3+3)=0$$

Left Hand Limit.

$$\lim_{x\to -3^-} f(x)$$
$$=\lim_{x\to -3^-}\frac{x^2-9}{x-3}$$
$$=\lim_{x\to -3^-}\frac{(x+3)(\cancel{x-3})}{(\cancel{x-3})}$$
$$=\lim_{x\to -3^-}(x+3)$$
$$(-3+3)=0$$

Tahir Mahmood
M.Sc. (Math)
Mob No: 03[illegible]610779

TAHIR

$\because \lim_{x\to -3^+} f(x) = \lim_{x\to -3^-} f(x)$

Thus Limit exists and is

$\lim_{x\to -3} f(x) = 0$

(iii) $f(x) = |x-5|$, $c=5$

Right Hand Limit	Left Hand Limit
$\lim_{x\to 5^+} f(x)$	$\lim_{x\to 5^-} f(x)$
$= \lim_{x\to 5^+} \lvert x-5\rvert$	$= \lim_{x\to 5^-} \lvert x-5\rvert$
$\because \lvert x-5\rvert > 0$ as $x>5$	$\lvert x-5\rvert < 0$ as $x<5$
$= \lim_{x\to 5^+} (x-5)$	$= \lim_{x\to 5^-} -(x-5)$
$= (5-5) = 0$	$= -(5-5) = 0$
$= \lim_{x\to 5^+} f(x) = 0$	$= \lim_{x\to 5^-} f(x) = 0$

$\because \lim_{x\to 5^+} f(x) = \lim_{x\to 5^-} f(x)$

Thus Limit exists and is

$\lim_{x\to 5} f(x) = 0$

Q.2 Discuss the Continuity at $x=c$.

(i) $f(x) = \begin{cases} 2x+5 & \text{if } x \le 2 \\ 4x+1 & \text{if } x > 2 \end{cases}$, $c=2$

Right Hand Limit	Left Hand Limit
$\lim_{x\to 2^+} f(x)$	$\lim_{x\to 2^-} f(x)$
$= \lim_{x\to 2^+} 4x+1$	$= \lim_{x\to 2^-} 2x+5$
$= 4(2)+1 = 8+1 = 9$	$= 2(2)+5 = 4+5 = 9$
$\lim_{x\to 2^+} f(x) = 9$	$\lim_{x\to 2^-} f(x) = 9$

$\because \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = 9$

So Limit exists and is $\lim_{x\to 2} f(x) = 9$

Now $f(x) = 2x+5$ for $c=2$

$f(2) = 2(2)+5 = 4+5 = 9$

$f(2) = 9$

$\because \lim_{x\to 2} f(x) = f(2) = 9$

Thus function is Continuous at $c=2$

(ii) $f(x) = \begin{cases} 3x-1 & \text{if } x<1 \\ 4 & \text{if } x=1 \\ 2x & \text{if } x>1 \end{cases}$, $c=1$

Right Hand Limit	Left Hand Limit
$\lim_{x\to 1^+} f(x)$	$\lim_{x\to 1^-} f(x)$
$= \lim_{x\to 1^+} 2x$	$= \lim_{x\to 1^-} 3x-1$
$= 2(1) = 2$	$= 3(1)-1 = 2$
$\lim_{x\to 1^+} f(x) = 2$	$\lim_{x\to 1^-} f(x) = 2$

$\because \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = 2$

Thus Limit exist and is $\lim_{x\to 1} f(x) = 2$

Now $f(1) = 4$

$\because f(1) \ne \lim_{x\to 1} f(x)$

Thus function is not Continuous at $c=1$

Q.3 If $f(x) = \begin{cases} 3x & \text{if } x \le -2 \\ x^2-1 & \text{if } -2<x<2 \\ 3 & \text{if } x \ge 2 \end{cases}$

Discuss continuity at $x=2$, $x=-2$

<u>Continuity at $x=2$</u>

Right Hand Limit	Left Hand Limit
$\lim_{x\to 2^+} f(x)$	$\lim_{x\to 2^-} f(x)$
$= \lim_{x\to 2^+} 3$	$= \lim_{x\to 2^-} x^2-1$
$= 3$	$= (2)^2-1 = 3$

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779

$\because \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = 3$

Thus Limit exists and is $\lim_{x\to 2} f(x) = 3$

Now $f(2) = 3$

Also $\lim_{x\to 2} f(x) = f(2) = 3$

Thus $f(x)$ is Continuous at $x = 2$

Continuouty at $x = -2$

Right Hand Limit	Left hand limit
$\lim_{x\to -2^+} f(x)$	$\lim_{x\to -2^-} f(x)$
$= \lim_{x\to -2^+} x^2-1$	$\lim_{x\to -2^-} 3x$
$= (-2)^2-1 = 4-1 = 3$	$= 3(-2) = -6$
$\lim_{x\to -2^+} f(x) = 3$	$\lim_{x\to -2} f(x) = -6$

$\because \lim_{x\to -2^+} f(x) \neq \lim_{x\to -2^-} f(x)$

so Limit does not exist and hence function is not Continuous at $x=-2$

Q.4 If $f(x) = \begin{cases} x+2 & \text{if } x \le -1 \\ c+2 & \text{if } x > -1 \end{cases}$ $c = ?$ so that $\lim_{x\to -1} f(x)$ exists.

Right Hand Limit	Left Hand Limit
$\lim_{x\to -1^+} f(x)$	$\lim_{x\to -1^-} f(x)$
$= \lim_{x\to -1^+} c+2$	$\lim_{x\to -1^-} x+2$
$= c+2$	$= (-1)+2 = 1$
$\lim_{x\to -1^+} f(x) = c+2$	$\lim_{x\to -1^-} f(x) = 1$

The Limit will exist if

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^-} f(x)$

$\Rightarrow \quad c+2 = 1$

$c = 1-2 \Rightarrow \boxed{c = -1}$

Q.5 Find the values of m and n so that function is continuous:

(i) $f(x) = \begin{cases} mx & \text{if } x < 3 \\ n & \text{if } x = 3 \\ -2x+9 & \text{if } x > 3 \end{cases}$

Right Hand Limit	Left Hand Limit
$\lim_{x\to 3^+} f(x)$	$\lim_{x\to 3^-} f(x)$
$= \lim_{x\to 3^+} (-2x+9)$	$\lim_{x\to 3^-} mx$
$= -2(3)+9 = -6+9 = 3$	$= m(3) = 3m$
$\lim_{x\to 3^+} f(x) = 3$	$\lim_{x\to 3^-} f(x) = 3m$

As function is continuous so limit must exist

$\Rightarrow \lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x)$

$3m = 3 \Rightarrow \boxed{m = 1}$

Thus $\lim_{x\to 3} f(x) = 3$

Also for continuous function

$\lim_{x\to 3} f(x) = f(3)$

$\boxed{3 = n}$

Thus $\boxed{m=1}$ and $\boxed{n=3}$

(iii) $f(x) = \begin{cases} mx & \text{if } x < 4 \\ x^2 & \text{if } x \ge 4 \end{cases}$

Right Hand Limit	Left Hand Limit
$\lim_{x\to 4^+} f(x)$	$\lim_{x\to 4^-} f(x)$
$= \lim_{x\to 4^+} x^2$	$= \lim_{x\to 4^-} mx$
$= (4)^2 = 16$	$= m(4) = 4m$

Tahir Mahmood
M.Sc. (Math)
Mob No: 0346-6510779

As function is Continuous so

Limit must exist.

$\Rightarrow \lim_{x\to 4^+} f(x) = \lim_{x\to 4^-} f(x)$

$16 = 4m \Rightarrow m = 16/4$

$\Rightarrow \boxed{m = 4}$

Q.6 If $f(x) = \begin{cases} \dfrac{\sqrt{2x+5} - \sqrt{x+7}}{(x-2)} & \text{if } x \neq 2 \\ K & \text{if } x = 2 \end{cases}$

Find k so that f is continuous at $x=2$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2}$$

$$= \lim_{x\to 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} \times \frac{\sqrt{2x+5} + \sqrt{x+7}}{\sqrt{2x+5} + \sqrt{x+7}}$$

$$= \lim_{x\to 2} \frac{(\sqrt{2x+5})^2 - (\sqrt{x+7})^2}{(x-2)[\sqrt{2x+5} + \sqrt{x+7}]}$$

$$= \lim_{x\to 2} \frac{2x+5-x-7}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})}$$

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{2x+5} + \sqrt{x+7})}$$

$$= \lim_{x\to 2} \frac{1}{\sqrt{2x+5} + \sqrt{x+7}}$$

$$= \frac{1}{\sqrt{2(2)+5} + \sqrt{2+7}}$$

$$= \frac{1}{\sqrt{4+5} + \sqrt{2+7}} = \frac{1}{\sqrt{9}+\sqrt{9}}$$

$$= \frac{1}{3+3} = \frac{1}{6}$$

$\lim_{x\to 2} f(x) = 1/6$

$f(2) = K$

As function is continuous at $x=2$ so

$\lim_{x\to 2} f(x) = f(2)$

$\boxed{K = 1/6}$

Thus for $K = 1/6$ f is Continuous.

Graph of a function:-

"The sketch of a function with the help of lines, points and Specific (مخصوص) interval on the paper or plane is called Graph."

Method to draw Graphs:

The following steps should be adopt to draw any graph:

* Take some specific values of x in the function.
* Calculate respective values of y
* Write interms of points.
* Take values of x along x-axis
* Take values of y along y-axis.
* Join every point without stopage and roughness.

Note: Graph must be smooth and consecutive line or curve.