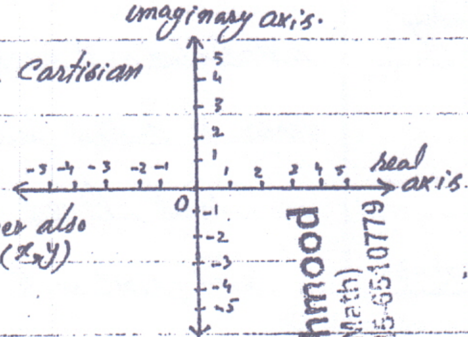
Geometrical Representation of Complex numbers on Complex Plane:

"The plane on which complex numbers are graphed is called Complex plane or Z plane".

In fact real plane and Complex plane are identical. Just the difference is the name of axes. In Complex plane x-axis is called real axis and y-axis is called imaginary axis.

$\because$ (x,y) is a Cartisian Point so $x+iy$ is a complex number also (x,y)



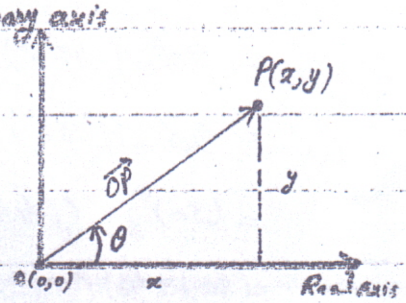
Complex plane.

Argand Diagram:-

"The figure on which one or more than complex numbers are represented is called Argand Diagram."

This idea was introduced by a French mathematician J.R Argand.

This diagram represents a Complex No. $P(x,y)$ as a point aswell as a Vector $\vec{OP}$



Modulus:-

"The distance of a Complex Point to the origion is called Modulus of Complex number."

If $P(x,y)$ is a Complex number then $|\vec{OP}| = \sqrt{x^2+y^2}$ is the modulus of complex number.

The modulus is usualy denoted by $|Z|$ or r

The Modulus is also called Absolute Value of Complex number."

Argument:-

The angle θ of a Complex number is known as Argument or amplitude of a complex number.

In the above diagram

$$\arg(z) = \theta = \tan^{-1}(y/x)$$

where $-\pi \le \theta \le \pi$

"The value of θ ($\arg(z)$) which satisfied the $\sin\theta$ and $\cos\theta$ is called Principal Argument."

Theorems:-

"The statement which is to prove and whose proof is true is called Theorem."

(1) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$ for all $z \in C$

Proof:-

Let $z = a+ib$ $\quad \bar{z} = a-ib$

$-z = -a-ib \quad -\bar{z} = -a+ib$

$|z| = \sqrt{a^2+b^2}$ ——— (1)

$|\bar{z}| = \sqrt{(a)^2+(-b)^2} = \sqrt{a^2+b^2}$ ——— (2)

$|-z| = \sqrt{(-a)^2+(-b)^2} = \sqrt{a^2+b^2}$ ——— (3)

$|-\bar{z}| = \sqrt{(-a)^2+(b)^2} = \sqrt{a^2+b^2}$ ——— (4)

Comparing (1), (2), (3), (4)

$|z| = |\bar{z}| = |-z| = |-\bar{z}|$ (Proved.)

(2) $\bar{\bar{z}} = z$ for all $z \in C$

Proof:- let $z = a+ib$ ——— (1)

$\bar{z} = a-ib$

$\bar{\bar{z}} = a+ib$ ——— (2)

So $\bar{\bar{z}} = z$ Proved

(3) $z\bar{z} = |z|^2$ for all $z \in C$

Proof:- Let $z = a+ib$

$\bar{z} = a-ib$

$z\bar{z} = (a+ib)(a-ib)$

$z\bar{z} = a.a + \cancel{ib.a} - \cancel{ib.a} - i^2 b.b$

$z\bar{z} = a^2 - i^2 b^2 \quad (\because i^2 = -1)$

$z\bar{z} = a^2+b^2$ ——— (1)

$|z| = \sqrt{a^2+b^2}$

$|z|^2 = a^2+b^2$ ——— (2)

Comparing (1) and (2)

$z\bar{z} = |z|^2$ (Proved)

(4) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$ for $z_1, z_2 \in C$

Proof:- Let $z_1 = a+ib$, $z_2 = c+id$

$\bar{z}_1 = a-ib \quad \bar{z}_2 = c-id$

$\because z_1+z_2 = (a+ib)+(c+id)$

$z_1+z_2 = (a+c)+i(b+d)$

$\overline{z_1+z_2} = (a+c) - i(b+d)$ ——— (1)

Now

$\bar{z}_1+\bar{z}_2 = (a-ib)+(c-id)$

$\bar{z}_1+\bar{z}_2 = (a+c) - ib - id$

$\bar{z}_1+\bar{z}_2 = (a+c) - i(b+d)$ ——— (2)

Comparing (1) and (2)

$\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$ (Proved)

(5) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ $z_2 \neq 0$ for $z_1, z_2 \in C$

Proof: Let $z_1 = a+ib$, $z_2 = c+id$

$\bar{z}_1 = a-ib$, $\bar{z}_2 = c-id$

$\frac{z_1}{z_2} = \frac{a+ib}{c+id}$

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$\frac{z_1}{z_2} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id}$

$\frac{z_1}{z_2} = \frac{ac - iad + ibc - i^2 bd}{(c)^2 - (id)^2}$

$\frac{z_1}{z_2} = \frac{ac - i(ad-bc) + bd}{c^2+d^2} \quad (\because i^2 = -1)$

$\frac{z_1}{z_2} = \frac{(ac+bd)}{c^2+d^2} - \frac{i(ad-bc)}{c^2+d^2}$

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Taking Conjugate on both sides.

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{(ac+bd)}{c^2+d^2} + i\frac{(ad-bc)}{c^2+d^2} \quad \text{—(1)}$$

Now

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{a-ib}{c-id}$$

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{a-ib}{c-id} \times \frac{c+id}{c+id}$$

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{ac-ibc+iad-i^2bd}{(c)^2-(id)^2} \quad (\because i^2=-1)$$

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{(ac+bd)+i(ad-bc)}{c^2+d^2}$$

$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{(ac+bd)}{c^2+d^2} + i\frac{(ad-bc)}{(c^2+d^2)} \quad \text{—(2)}$$

Comparing (1) and (2)

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \quad \text{(Proved)}$$

(6) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ for all $z_1, z_2 \in C$

Proof:- Let $z_1 = a+ib$ $\quad z_2 = c+id$

$|z_1| = \sqrt{a^2+b^2}$ $\quad |z_2| = \sqrt{c^2+d^2}$

$$|z_1 \cdot z_2| = |(a+ib)\cdot(c+id)|$$

$$= |ac+ibc+ida+i^2bd|$$

$$= |ac+i(ad+bc)-bd| \quad (\because i^2=-1)$$

$$= |(ac-bd)+i(ad+bc)|$$

$$= \sqrt{(ac-bd)^2+(ad+bc)^2}$$

$$= \sqrt{a^2c^2+b^2d^2-2abcd+a^2d^2+b^2c^2+2abcd}$$

$$= \sqrt{a^2c^2+a^2d^2+b^2d^2+b^2c^2}$$

$$= \sqrt{a^2(c^2+d^2)+b^2(c^2+d^2)}$$

$$= \sqrt{(a^2+b^2)(c^2+d^2)} \quad \text{—(1)}$$

Now

$$|z_1| \cdot |z_2| = \sqrt{a^2+b^2} \cdot \sqrt{c^2+d^2}$$

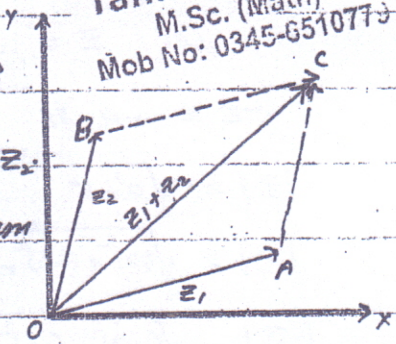
$$|z_1| \cdot |z_2| = \sqrt{(a^2+b^2)(c^2+d^2)} \quad \text{—(2)}$$

Comparing (1) and (2)

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2| \quad \text{(Proved).}$$

(7) $|z_1| - |z_2| \le |z_1+z_2| \le |z_1| + |z_2|$

Proof:-



Let $\overline{OA}$ and $\overline{OB}$ represents z_1 & z_2 and $\overline{OC}$ is the sum of $\overline{OA}$ and $\overline{OB}$

$\because OC = z_1+z_2$

On completing the parallelogram OACB having $\overline{OA}$ and $\overline{OB}$ adjacent sides, we get

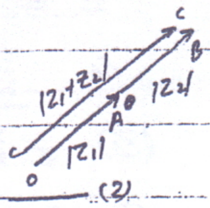
$|z_1| = \overline{OA}$ $\quad |z_2| = \overline{OB}$ $\quad |z_1+z_2| = \overline{OC}$

In the ΔOAC

$$OA + AC > OC \quad (\because AC = OB)$$

$$|z_1+z_2| < |z_1| + |z_2| \quad \text{—(1)}$$

We take these lines $\overline{OA}$ and $\overline{OB}$ will be parallel then



$$\overline{OC} = \overline{OA} + \overline{OB}$$

$$|z_1+z_2| = |z_1| + |z_2| \quad \text{—(2)}$$

From (1) and (2)

$$|z_1+z_2| \le |z_1| + |z_2| \quad \text{—(A)}$$

(This is called triangular inequality)

Also in ΔOAC

$$OC + CA > OA \quad \text{and} \quad CO + OA > AC$$

$$|z_1+z_2| + |z_2| > |z_1| \quad \text{and} \quad |z_1+z_2| + |z_1| > |z_2|$$

$$|z_1+z_2| > |z_1| - |z_2| \quad \text{and} \quad |z_1| - |z_2| > -|z_1+z_2|$$

We can write

$|z_1| - |z_2| < |z_1 + z_2|$ and $-|z_1 + z_2| < |z_1| - |z_2|$

using the result $|x| \le a \Rightarrow -a \le x \le a$

$-|z_1 + z_2| < |z_1| - |z_2| < |z_1 + z_2|$

$||z_1| - |z_2|| < |z_1 + z_2|$

The equality will be hold if the point A and C become collinear to B.

So

$||z_1| - |z_2|| \le |z_1 + z_2|$ ——— (B)

Combining (A) and (B)

$||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$

(Proved).

Polar Form of a Complex Number:

For a complex number

$z = x + iy$

The polar Co-ordinates are

$x = r\cos\theta \qquad y = r\sin\theta$

$|z| = \sqrt{x^2 + y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$

$|z| = \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = \sqrt{r^2(1)}$

$|z| = r$

$\because z = x + iy$

$z = r\cos\theta + ir\sin\theta$

$z = r(\cos\theta + i\sin\theta)$

For our Convenience (آسانی), we can write

$z = r\,\text{Cis}\,\theta$.

De-Moivre's Theorem:-

If n is an integer Then

$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

This theorem was introduced by a french mathematician Abraham De Moivre's

Useful Results:-

(i) $z_1 z_2 = r_1 r_2\left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)$

(ii) $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$

(iii) $\left(\frac{z_1}{z_2}\right)^n = \frac{r_1^n}{r_2^n}\left[\cos n(\theta_1 - \theta_2) + i\sin n(\theta_1 - \theta_2)\right]$

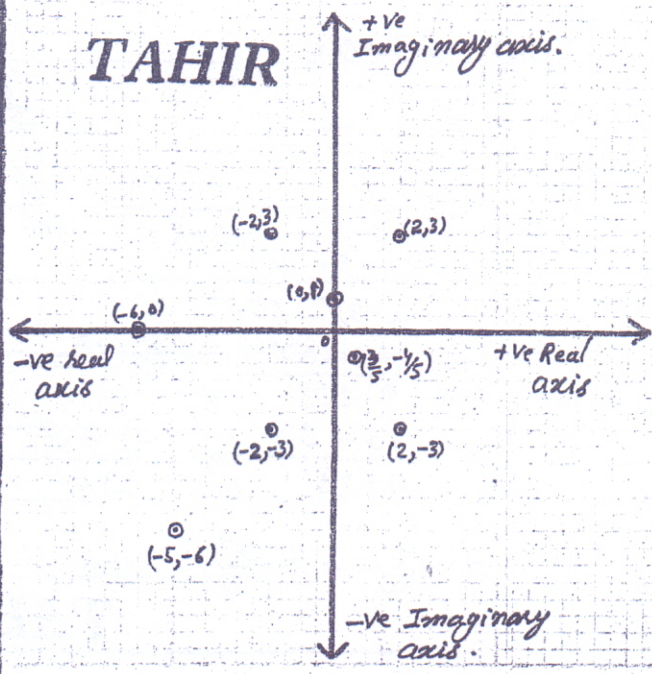
(iv) $(z_1 z_2)^n = r_1^n r_2^n\left[\cos n(\theta_1 + \theta_2) + i\sin n(\theta_1 + \theta_2)\right]$

Exercise: 1.3

Q.1 This Question is just to fix the given Complex Numbers on the Graph.

Scale 2 Small Horizontal Sq = 1 unit

2 Small Verticle Sq. = 1 unit.



Q.2 Find Multiplicative inverse :-

(i) $-3i \Rightarrow 0-3i$

$\because \frac{1}{a}$ is the multipl. inverse of a

$\because \frac{1}{0-3i}$

$= \frac{1}{0-3i} \times \frac{0+3i}{0+3i}$

$= \frac{0+3i}{0^2-(3i)^2}$

$= \frac{0+3i}{0+9} \Rightarrow \frac{3}{9}i$

$M.I = \frac{1}{3}i$ Ans.

(ii) $1-2i$

$M.I = \frac{1}{1-2i}$

$= \frac{1}{1-2i} \times \frac{1+2i}{1+2i}$

$= \frac{1+2i}{(1)^2-(2i)^2} \Rightarrow \frac{1+2i}{1-4i^2}$

$= \frac{1+2i}{1+4}$ $(\because i^2=-1)$

$= \frac{1+2i}{5} \Rightarrow \frac{1}{5}+i\frac{2}{5}$ Ans.

(iii) $-3-5i$

$M.I. = \frac{1}{-3-5i}$

$= \frac{1}{-3-5i} \times \frac{-3+5i}{-3+5i}$

$= \frac{-3+5i}{(-3)^2-(5i)^2}$

$= \frac{-3+5i}{9-i^2 25}$ $(\because i^2=-1)$

$= \frac{-3+5i}{9+25} \Rightarrow \frac{-3+5i}{34}$

$= \frac{-3}{34}+i\frac{5}{34}$ Ans.

(iv) $(1,2) \Rightarrow 1+2i$

Multip. Inverse $= \frac{1}{1+2i}$

$= \frac{1}{1+2i} \times \frac{1-2i}{1-2i}$

$= \frac{1-2i}{(1)^2-(2i)^2}$

$= \frac{1-2i}{1-4i^2}$ $(\because i^2=-1)$

$= \frac{1-2i}{1+4} = \frac{1-2i}{5}$

$= \frac{1}{5}-\frac{2}{5}i$ Ans.

Q3 Simplify :-

(i) i^{101}

$\because i^2=-1$

$\because i^4=1$

$= (i^4)^{25} \cdot i$

$= (1)^{25} \cdot i \Rightarrow 1 \cdot i$

$= i$ Ans.

(ii) $(-ai)^4$ $a \in \mathbb{R}$

$= a^4 i^4$

$= a^4(1)$ $(\because i^4=1)$

$= a^4$ Ans

(iii) i^{-3}

$= \frac{1}{i^3} \Rightarrow \frac{1}{i^2 \cdot i}$ $(\because i^2=-1)$

$= \frac{1}{(-1)i} \Rightarrow \frac{-1}{i}$ Ans.

(iv) i^{-10}

$= \frac{1}{i^{10}} \Rightarrow \frac{1}{(i^4)^2 \cdot i^2}$

$\because i^2=-1$

$\because i^4=1$

$= \frac{1}{(1)^4(-1)}$

$= \frac{1}{1(-1)} = -1$ Ans.

Q.4 Prove $\bar{z} = z$ iff z is real.

Let $z = a+ib$

$\because z = \bar{z}$

$a+ib = a-ib$

$a+ib-a+ib=0$

$2ib=0 \qquad 2i\neq 0$

$b=0$ So z is real.

Conversely if z is real then $b=0$

$z = a+0i = a$ —(1)

$\bar{z} = a-i0 = a$ —(2)

From (1) and (2)

$\bar{z} = z$ (Prove).

Q.5 Simplify in the form of $a+ib$

(i) $5+2\sqrt{-4}$

$=5+2\sqrt{-1\times 4}$

$=5+2\sqrt{4}\cdot\sqrt{-1} \qquad (\because i=\sqrt{-1})$

$=5+2\sqrt{4}\, i$

$=5+2.2i \Rightarrow 5+4i$ Ans.

(ii) $(2+\sqrt{-3})(3+\sqrt{-3})$

$=(2+\sqrt{-1\times3})(3+\sqrt{-1\times3}) \qquad (\because i=\sqrt{-1})$

$=(2+\sqrt{3}\, i)(3+\sqrt{3}\, i)$

$=6+2\sqrt{3}i+3\sqrt{3}i+\sqrt{3}i\cdot\sqrt{3}i$

$=6+(2+3)\sqrt{3}i+3i^2 \qquad \because i^2=-1$

$=6-3+5\sqrt{3}i$

$=3+5\sqrt{3}i$ Ans.

(iii) $\dfrac{2}{\sqrt{5}+\sqrt{-8}}$

$=\dfrac{2}{\sqrt{5}+\sqrt{-1\times8}} \qquad (\because i=\sqrt{-1})$

$=\dfrac{2}{\sqrt{5}+\sqrt{8}i}\Rightarrow\dfrac{2}{\sqrt{5}+2\sqrt{2}i} \qquad (\because\sqrt{8}=2\sqrt{2})$

$=\dfrac{2}{\sqrt{5}+2\sqrt{2}i}\times\dfrac{\sqrt{5}-2\sqrt{2}i}{\sqrt{5}-2\sqrt{2}i}$

$=\dfrac{2(\sqrt{5}-2\sqrt{2}i)}{(\sqrt{5})^2-(2\sqrt{2})^2}$

$=\dfrac{2\sqrt{5}-4\sqrt{2}i}{5-8i^2} \qquad (\because i^2=-1)$

$=\dfrac{2\sqrt{5}-4\sqrt{2}i}{5+8}\Rightarrow\dfrac{2\sqrt{5}-4\sqrt{2}i}{13}$

$=\dfrac{2\sqrt{5}}{13}-i\dfrac{4\sqrt{2}}{13}$ Ans.

(iv) $\dfrac{3}{\sqrt{6}-\sqrt{-12}}$

$=\dfrac{3}{\sqrt{6}-\sqrt{-1\times12}} \qquad (\because i=\sqrt{-1})$

$=\dfrac{3}{\sqrt{6}-\sqrt{12}i}$

$=\dfrac{3}{\sqrt{6}-\sqrt{12}i}\times\dfrac{\sqrt{6}+\sqrt{12}i}{\sqrt{6}+\sqrt{12}i}$

$=\dfrac{3(\sqrt{6}+\sqrt{4\times3}i)}{(\sqrt{6})^2-(\sqrt{12}i)^2}$

$=\dfrac{3\sqrt{6}+3.2\sqrt{3}i}{6-12i^2} \qquad (\because i^2=-1)$

$=\dfrac{3\sqrt{6}+6\sqrt{3}i}{6+12}\Rightarrow\dfrac{3\sqrt{6}+6\sqrt{3}i}{18}$

$=\dfrac{{}^{1}\cancel{3}\sqrt{6}}{\cancel{18}\,6}+i\dfrac{\cancel{6}\sqrt{3}}{\cancel{18}\,3}$

$=\dfrac{\sqrt{6}}{6}+i\dfrac{\sqrt{3}}{3} \qquad (\because a = \dfrac{\sqrt{a}}{a})$

$=\dfrac{1}{\sqrt{6}}+i\dfrac{1}{\sqrt{3}}$ Ans.

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Q.6 Show that $\forall\ z \in C$

(i) $z^2 + \bar{z}^2$ is real.

Soln:- Let $z = x+iy$

$\bar{z} = x-iy$

$\because z^2+\bar{z}^2 = (x+iy)^2 + (x-iy)^2$

$z^2+\bar{z}^2 = x^2+i^2y^2+\cancel{2ixy}+x^2+i^2y^2-\cancel{2ixy}$

$z^2+\bar{z}^2 = x^2-y^2+x^2-y^2 \quad (\because i^2=-1)$

$z^2+\bar{z}^2 = 2x^2-2y^2$

$\bar{z}^2+z^2 = 2(x^2-y^2)$

$z^2+\bar{z}^2$ is a real number because no i is involve in answer.

(ii) $(z-\bar{z})^2$ is real.

Soln:- Let $z = x+iy$

$\bar{z} = x-iy$

$\because (z-\bar{z})^2 = \{(x+iy)-(x-iy)\}^2$

$(z-\bar{z})^2 = \{x+iy-x+iy\}^2$

$(z-\bar{z})^2 = \{2iy\}^2$

$(z-\bar{z})^2 = 4i^2y^2 \quad \because i^2 = (-1)$

$(z-\bar{z})^2 = -4y^2$

This shows that $(z-\bar{z})^2$ is a real as there is not term of iota (i) involved in it.

Q.7 Simplify the following:

(i) $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3$

$r\cos\theta = -\frac{1}{2} \qquad r\sin\theta = \frac{\sqrt{3}}{2}$

$r^2\cos^2\theta = \frac{1}{4} \qquad r^2\sin^2\theta = \frac{3}{4}$

$r^2\cos^2\theta + r^2\sin^2\theta = \frac{1}{4}+\frac{3}{4}$

$r^2(1) = \frac{4}{4} \implies r^2 = 1$

$r = 1$

$\tan\theta = \frac{r\sin\theta}{r\cos\theta} = \frac{\sqrt{3}/2}{-1/2}$

$\tan\theta = -\sqrt{3}$

$\theta = \tan^{-1}(\sqrt{3})$

$\theta = 120^\circ$

$\therefore \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3 = \left[1\left(\cos(120^\circ)+i\sin 120^\circ\right)\right]^3$

$= \left(\cos(3\times120^\circ)+i\sin(3\times120^\circ)\right)$

$= (\cos 360^\circ + i\sin 360^\circ)$

$= (1+i0)$

$= 1$ Ans.

(ii) $\left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right)^3$

$r\cos\theta = -\frac{1}{2}$ $\quad r\sin\theta = -\frac{\sqrt{3}}{2}$

$r^2\cos^2\theta = \frac{1}{4}$ $\quad r^2\sin^2\theta = \frac{3}{4}$

$r^2\cos^2\theta + r^2\sin^2\theta = \frac{1}{4} + \frac{3}{4}$

$r^2(1) = \frac{4}{4} \Rightarrow r^2 = 1$

$r = 1$

$\tan\theta = \frac{r\sin\theta}{r\cos\theta} = \frac{-\sqrt{3}/2}{-1/2}$

$\tan\theta = \sqrt{3} \Rightarrow \theta = \tan^{-1}(\sqrt{3})$

$\theta = 60°, 240°$

$\because \left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right)$ is in 3rd Quadrent so

$\theta = 240°$

$\therefore \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right)^3 = \left[1(\cos 240° + i\sin 240°)\right]^3$

$= \cos(3\times240°) + i\sin(3\times240°)$

$= \cos 720° + i\sin 720°$

$= (1 + i0) = 1$ Ans.

(iii) $\left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right)^{-2} \cdot \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= \left(\frac{-1-\sqrt{3}i}{2}\right)^{-2}\left(\frac{-1-\sqrt{3}i}{2}\right)$

$= \left(\frac{2}{-1-\sqrt{3}i}\right)^2 \cdot \left(\frac{-1-\sqrt{3}i}{2}\right)$

$= \frac{2^{\cancel{2}}}{(-1-\sqrt{3}i)^{\cancel{2}}} \times \frac{\cancel{(-1-\sqrt{3}i)}}{\cancel{2}}$

$= \frac{2}{-1-\sqrt{3}i}$

$= \frac{2}{-1-\sqrt{3}i} \times \frac{-1+\sqrt{3}i}{-1+\sqrt{3}i}$

$= \frac{2(-1+\sqrt{3}i)}{(-1)^2 - (\sqrt{3}i)^2}$

$= \frac{2(-1+\sqrt{3}i)}{1-3i^2}$ $\quad (\because i^2 = -1)$

$= \frac{2(-1+\sqrt{3}i)}{1+3} \Rightarrow \frac{\cancel{2}(-1+\sqrt{3}i)}{\cancel{4}2}$

$= \frac{-1+\sqrt{3}i}{2}$ Ans.

(iv) $(a+ib)^2$

$= a^2 + i^2b^2 + 2iab$

$= a^2 - b^2 + 2abi$ $\quad (\because i^2 = -1)$

$= a^2 + 2iab - b^2$ Ans.

$= (a^2-b^2) + i(2ab).$

(v) $(a+ib)^{-2}$

$= \frac{1}{(a+ib)^2} \Rightarrow \frac{1}{a^2+i^2b^2+2iab}$

$= \frac{1}{(a^2-b^2)+2iab}$ $\quad (\because i^2 = -1)$

$= \frac{1}{(a^2-b^2)+2iab} \times \frac{(a^2-b^2)-2iab}{(a^2-b^2)-2iab}$

$= \frac{(a^2-b^2)-2iab}{(a^2-b^2)^2-(2iab)^2}$

$= \frac{(a^2-b^2)-2iab}{a^4+b^4-2a^2b^2-4i^2a^2b^2}$

$= \frac{(a^2-b^2)-2iab}{a^4+b^4-2a^2b^2+4a^2b^2}$ $\quad (\because i^2 = -1)$

$= \frac{(a^2-b^2)-2iab}{a^4+b^4+2a^2b^2} \Rightarrow \frac{(a^2-b^2)-2iab}{(a^2+b^2)^2}$

$= \frac{(a^2-b^2)}{(a^2+b^2)^2} - i\frac{2ab}{(a^2+b^2)^2}$ Ans.

(vi) $(a+ib)^3$

$= a^3+(ib)^3+3aib(a+ib)$

$= a^3+i^3b^3+3a^2ib+3ai^2b^2$

$= a^3+i^2ib^3+3i\,a^2b+3ab^2i^2$ $\quad (\because i^2=-1)$

$= a^3-ib^3+3ia^2b-3ab^2$

$= a^3-3ab^2-ib^3+3ia^2b$

$= a(a^2-3b^2)-i(b^3-3a^2b)$ Ans.

(vii) $(a-bi)^3$

$= a^3-(ib)^3-3aib(a-ib)$

$= a^3-i^3b^3-3a^2ib+3ai^2b^2$

$= a^3-i^2ib^3-3a^2ib+3ai^2b^2$ $\quad (\because i^2=-1)$

$= a^3-(-1)ib^3-3a^2ib+3a(-1)b^2$

$= a^3+ib^3-3a^2ib-3ab^2$

$= (a^3-3ab^2)+i(b^3-3a^2b)$ Ans.

(viii) $(3-\sqrt{-4})^{-3} \Rightarrow (3-\sqrt{-1\times 4})^{-3}$

$= (3-\sqrt{4}\,i)^{-3} \Rightarrow (3-2i)^{-3}$

$= \dfrac{1}{(3-2i)^3}$

$= \dfrac{1}{(3-2i)^3} \times \dfrac{(3+2i)^3}{(3+2i)^3}$

$= \dfrac{(3+2i)^3}{(3-2i)^3\cdot(3+2i)^3}$

$= \dfrac{(3)^3+(2i)^3+3(2i)(3)[3+2i]}{[(3-2i)(3+2i)]^3}$

$= \dfrac{27+8i^3+54i+36i^2}{[(3)^2-(2i)^2]^3}$

$= \dfrac{27+i^2\cdot i\,8+54i+36i^2}{(9-4i^2)^3}$ $\quad (\because i^2=-1)$

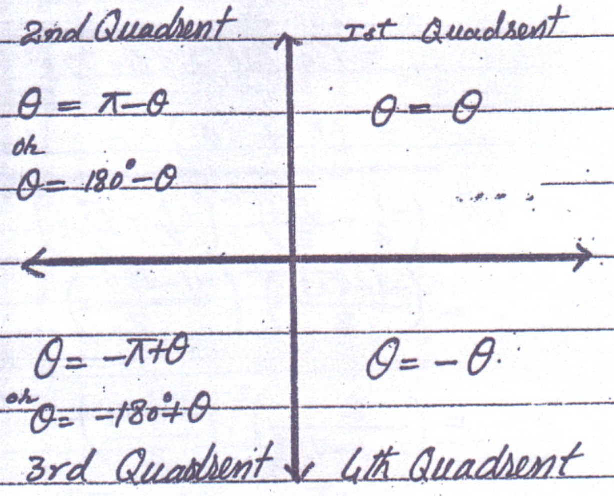
$= \dfrac{27+(-1)i8+54i-36}{(9+4)^3}$

$= \dfrac{(27-36)-8i+54i}{(13)^3}$

$= \dfrac{-9+46i}{2197}$

$= \dfrac{-9}{2197}+i\dfrac{46}{2197}$ Ans.

Note: we can take argument θ as given below.



Alternatively:-

Q.7 (i) $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3$

$\because (a+b)^3 = a^3+3a^2b+3ab^2+b^3$

So

$$\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3 = \left(-\frac{1}{2}\right)^3+3\left(-\frac{1}{2}\right)^2\left(\frac{\sqrt{3}}{2}i\right)+3\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}i\right)^2+\left(\frac{\sqrt{3}}{2}i\right)^3$$

$$= -\frac{1}{8}+3\left(\frac{1}{4}\right)\left(\frac{\sqrt{3}}{2}i\right)-3\left(\frac{1}{2}\right)\left(\frac{3}{4}i^2\right)+\frac{3\sqrt{3}}{8}i^3$$

$$= -\frac{1}{8}+\frac{3\sqrt{3}}{8}i-\frac{9}{8}i^2+\frac{3\sqrt{3}}{8}i^3$$

$$= -\frac{1}{8}+\frac{3\sqrt{3}}{8}i-\frac{9}{8}(-1)+\frac{3\sqrt{3}}{8}(-i) \qquad \because i^2=-1, \ \because i^3=-i$$

$$= -\frac{1}{8}+\cancel{\frac{3\sqrt{3}}{8}i}+\frac{9}{8}-\cancel{\frac{3\sqrt{3}}{8}i}$$

$$= \frac{9}{8}-\frac{1}{8} = \frac{9-1}{8} = \frac{8}{8} = 1$$

Thus $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3 = 1+0i$ Ans.

Q.7 (ii) $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^3$

$\because (a-b)^3 = a^3-3a^2b+3ab^2-b^3$

So

$$\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^3 = \left(-\frac{1}{2}\right)^3-3\left(-\frac{1}{2}\right)^2\left(\frac{\sqrt{3}}{2}i\right)+3\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}i\right)^2-\left(\frac{\sqrt{3}}{2}i\right)^3$$

$$= -\frac{1}{8}-3\left(\frac{1}{4}\right)\left(\frac{\sqrt{3}}{2}i\right)-\frac{3}{2}\left(\frac{3}{4}i^2\right)-\frac{3\sqrt{3}}{8}i^3$$

$$= -\frac{1}{8}-\frac{3\sqrt{3}}{8}i-\frac{9}{8}i^2-\frac{3\sqrt{3}}{8}i^3$$

$$= -\frac{1}{8}-\frac{3\sqrt{3}}{8}i-\frac{9}{8}(-1)-\frac{3\sqrt{3}}{8}(-i) \qquad \because i^2=-1, \ \because i^3=-i$$

$$= -\frac{1}{8}-\cancel{\frac{3\sqrt{3}}{8}i}+\frac{9}{8}+\cancel{\frac{3\sqrt{3}}{8}i}$$

$$= \frac{9}{8}-\frac{1}{8} = \frac{9-1}{8} = \frac{8}{8} = 1$$

Thus $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^3 = 1+0i$ Ans.

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The End