(iv) $f(x) = (x-5)^{1/2} \quad x \geq 5$

$f(x) = \sqrt{x-5} \quad x \geq 5$

Thus $f(x)$ has real value for $x \geq 5$ so $Dom(f) = [5, \infty[$

$Range(f) = [0, \infty[$

Now $Dom(f^{-1}) = [0, \infty[$

$Ran(f^{-1}) = [5, \infty[$

Limit of a Function:-

The concept of Limit is a basic tool of Calculus.

"Let $f(x)$ be a real valued function and $x \to a$ (x approches to a) then L (specific number) is called it limit and is denoted as

$$\lim_{x \to a} f(x) = L$$

* $x \to a$ means $x \neq a$, "x" is going towards "a".

* Limit of any function exists if L is a definate number.

* To apply Limit simplify the function

* Incase of $x \to \infty$ divided the nominator and denominator by the highest power of variable involved in $f(x)$.

Some Important Proofs:-

(1) Prove $\lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}$

Let $x = a+h$ and $\lim_{h \to 0}$

LHS. $\lim_{h \to 0} \frac{(a+h)^n - a^n}{a+h-a}$

$\lim_{h \to 0} \frac{(a+h)^n - a^n}{h}$

$= \lim_{h \to 0} \frac{a^n(1+\frac{h}{a})^n - a^n}{h}$ (using Binomial Series)

$= \lim_{h \to 0} \frac{a^n\left\{1 + \frac{nh}{a} + \frac{n(n-1)}{2!}\frac{h^2}{n^2} + \cdots\right\} - a^n}{h}$

$= \lim_{h \to 0} \frac{a^n + \frac{nh}{a}a^n + \frac{n(n-1)}{2!}\frac{h^2}{n^2}a^n + \cdots - a^n}{h}$

$= \lim_{h \to 0} \frac{nha^{n-1} + \frac{n(n-1)}{2!}h^2a^{n-2} + \cdots}{h}$

$= \lim_{h \to 0} \frac{h\left\{na^{n-1} + \frac{n(n-1)}{2!}ha^{n-2} + \cdots\right\}}{h}$

$= \lim_{h \to 0} na^{n-1} + \frac{n(n-1)}{2!}ha^{n-2} + \cdots$

By Applying Limit

$= na^{n-1} + \frac{n(n-1)}{2!}(0)a^{n-2} + \cdots$

$= na^{n-1} + 0 = na^{n-1}$

Thus $\lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}$ (Proved)

* $\lim_{x \to 0} 1/x = \infty$

* $\lim_{x \to \infty} 1/x = 0$

* $\lim_{x \to -\infty} 1/x = 0$

(2) Prove $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ where $e = 2.7183\ldots$

LHS. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ ... Using binomial Series.

$$= \lim_{n\to\infty}\left\{1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \cdots\cdots\right\}$$

$$= \lim_{n\to\infty}\left\{1+1+\frac{1}{2!}\frac{n-1}{n} + \frac{1}{3!}\frac{n-1}{n}\cdot\frac{n-2}{n} + \cdots\cdots\right\}$$

$$= \lim_{n\to\infty}\left\{2+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots\cdots\right\}$$

Applying Limit, we have

$$= 2+\frac{1}{2!}(1-0)+\frac{1}{3!}(1-0)(1-0)+\cdots\cdots$$

$$= 2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots\cdots = 2+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}.$$

$$= 2+0.5+0.16667+0.04167+\cdots\cdots = 2.7183$$

$$= e$$

Thus $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ (Proved).

(3) Prove $\lim_{n\to 0}(1+n)^{1/n} = e$

Let $n = \frac{1}{x} \Rightarrow x = \frac{1}{n}$ as $n\to 0$ $x\to\infty$

$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ (Using above theorem)

(4) Prove $\lim_{x\to 0}\frac{a^x-1}{x} = \log_e a = \ln a$

LHS $= \lim_{x\to 0}\frac{a^x-1}{x}$

Let $a^x-1 = y \Rightarrow a^x = 1+y$

By taking ln on both Sides $\ln a^x = \ln(1+y)$

$x\ln a = \ln(1+y) \Longrightarrow x = \frac{\ln(1+y)}{\ln a}$

As $x\to 0$ $y\to 0$

$$\lim_{y\to 0}\frac{y}{\frac{\ln(1+y)}{\ln a}} \Longrightarrow \lim_{y\to 0}\frac{\ln a}{\frac{1}{y}\ln(1+y)}$$

$$= \lim_{y\to 0}\frac{\ln a}{\ln(1+y)^{1/y}} \Rightarrow \lim_{y\to 0}\frac{\ln a}{\ln e}$$

$$= \ln a \quad \text{(Proved)}$$

$\because \log_a m^n = n\log_a m$

$\lim_{y\to 0}(1+y)^{1/y} = e$

$\because \ln e = 1$

(5) Prove $\lim_{x\to 0}\frac{e^x-1}{x}=1$

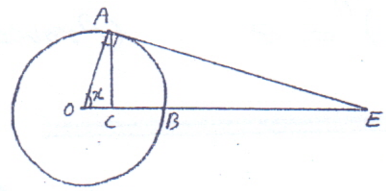
$\because \lim_{x\to 0}\frac{a^x-1}{x} = \ln a$

$\lim_{x\to 0}\frac{e^x-1}{x} = \ln e$

$\Rightarrow \lim_{x\to 0}\frac{e^x-1}{x} = 1 \quad \because \ln e = 1$ (Proved)

(6) Prove $\lim_{x\to 0}\frac{\sin x}{x} = 1$

Proof:- Let a unit circle having $\widehat{AB}$ and $\angle AOB = x$.



Draw $AC \perp$ to OB and tangent at A. Join OB and AE.

From figure

$|AE| > \widehat{AB} > |AC|$

Dividing by $|OA|$, we have

$\frac{|AE|}{|OA|} > \frac{\widehat{AB}}{|OA|} > \frac{|AC|}{|OA|}$

From ΔAOC $\quad \sin x = \frac{|AC|}{|OA|}$

From ΔAOE $\quad \tan x = \frac{|AE|}{|OA|}$

From ABO sector $\quad \widehat{AB} = |OA| \cdot x$

$\Rightarrow \widehat{AB} = x \quad \because |OA| = 1$

Thus $\tan x > x > \sin x$

Dividing by $\sin x$, we have

$\frac{\tan x}{\sin x} > \frac{x}{\sin x} > \frac{\sin x}{\sin x}$

$\frac{1}{\cos x} > \frac{x}{\sin x} > 1$

Inverting throughout

$\cos x < \frac{\sin x}{x} < 1$

Applying $\lim_{x\to 0}$, we have

$\lim_{x\to 0}\cos x < \lim_{x\to 0}\frac{\sin x}{x} < 1$

$1 < \lim_{x\to 0}\frac{\sin x}{x} < 1$

$\Rightarrow \lim_{x\to 0}\frac{\sin x}{x} = 1$ (Proved)

Exercise:1.3

Q.1 Evaluate the limits:

(i) $\lim_{x\to 3}(2x+4)$

$= \lim_{x\to 3} 2x + \lim_{x\to 3} 4$

$= 2\lim_{x\to 3} x + 4$

$= 2(3) + 4 = 6+4 = 10$

$\lim_{x\to 3}(2x+4) = 10$ Ans.

(ii) $\lim_{x\to 1} 3x^2-2x+4$

$= 3\lim_{x\to 1} x^2 - 2\lim_{x\to 1} x + 4$

$= 3(1)^2 - 2(1) + 4 = 3-2+4$

$= 5$

Thus $\lim_{x\to 1}(3x^2-2x+4) = 5$ (Ans.

(iii) $\lim_{x\to 3}\sqrt{x^2+x+4}$

$= \lim_{x\to 3}(x^2+x+4)^{1/2}$

$= \left[\lim_{x\to 3} x^2 + \lim_{x\to 3} x + 4\right]^{1/2}$

$= \left[(3)^2+(3)+4\right]^{1/2} = (9+3+4)^{1/2} = (16)^{1/2}$

$= 4 \Rightarrow \lim_{x\to 3}\sqrt{x^2+x+4} = 4$ Ans.

(iv) $\lim\limits_{x\to 2} x\sqrt{x^2-4}$

$= \lim\limits_{x\to 2} x \cdot \lim\limits_{x\to 2}\sqrt{x^2-4}$

$= (2)\cdot\sqrt{(2)^2-4}$

$= 2\cdot(0) = 0$

Thus $\lim\limits_{x\to 2} x\sqrt{x^2-4} = 0$ Ans.

(v) $\lim\limits_{x\to 2}\left(\sqrt{x^3+1}-\sqrt{x^2+5}\right)$

$= \lim\limits_{x\to 2}\sqrt{x^3+1} - \lim\limits_{x\to 2}\sqrt{x^2+5}$

$= \sqrt{(2)^3+1} - \sqrt{(2)^2+5}$

$= \sqrt{9}-\sqrt{9} = 0$

Thus $\lim\limits_{x\to 2}\left(\sqrt{x^3+1}-\sqrt{x^2+5}\right) = 0$ Ans.

(vi) $\lim\limits_{x\to -2}\dfrac{2x^3+5x}{3x-2}$

$= \lim\limits_{x\to -2}\dfrac{2x^3}{3x-2} + \lim\limits_{x\to -2}\dfrac{5x}{3x-2}$

$= \dfrac{2(-2)^3}{3(-2)-2} + \dfrac{5(-2)}{3(-2)-2}$

$= \dfrac{-16}{-6-2} + \dfrac{-10}{-6-2} = \dfrac{-16}{-8} + \dfrac{-10}{-8}$

$= 2+\dfrac{5}{4} = \dfrac{8+5}{4} = \dfrac{13}{4}$

$\lim\limits_{x\to -2}\dfrac{2x^3+5x}{3x-2} = 13/4$ Ans.

Q.2 EVALUATE The LIMITS:-

(i) $\lim\limits_{x\to -1}\dfrac{x^3-x}{x+1}$

$= \lim\limits_{x\to -1}\dfrac{x(x^2-1)}{x+1}$

$= \lim\limits_{x\to -1}\dfrac{x\cdot(x+1)(x-1)}{(x+1)}$

$\because a^2-b^2=(a+b)(a-b)$

$= \lim\limits_{x\to -1} x \cdot \lim\limits_{x\to -1} x-1$

$= (-1)\cdot(-1-1)$

$= (-1)\cdot(-2) = 2$

$\lim\limits_{x\to -1}\dfrac{x^3-x}{x+1} = 2$ Ans.

(ii) $\lim\limits_{x\to 1}\dfrac{3x^3+4x}{x^2+x}$

$= \lim\limits_{x\to 1}\dfrac{x(3x^2+4)}{x(x+1)}$

$= \lim\limits_{x\to 1}\dfrac{3x^2+4}{x+1}$

$= \lim\limits_{x\to 1}\dfrac{3x^2}{x+1} + \lim\limits_{x\to 1}\dfrac{4}{x+1}$

$= \dfrac{3(1)^2}{1+1} + \dfrac{4}{1+1}$

$= \dfrac{3}{2}+\dfrac{4}{2} = \dfrac{3+4}{2} = 7/2$

Thus $\lim\limits_{x\to 1}\dfrac{3x^3+4x}{x^2+x} = 7/2$ Ans.

(iii) $\lim\limits_{x\to 2}\dfrac{x^3-8}{x^2+x-6}$

$= \lim\limits_{x\to 2}\dfrac{(x)^3-(2)^3}{x^2+3x-2x-6}$

$= \lim\limits_{x\to 2}\dfrac{\cancel{(x-2)}(x^2+2x+4)}{(x+3)\cancel{(x-2)}}$

$= \lim\limits_{x\to 2}\dfrac{x^2+2x+4}{x+3}$

$= \dfrac{(2)^2+2(2)+4}{(2)+3} = \dfrac{4+4+4}{5}$

$= \dfrac{12}{5}$

Thus $\lim\limits_{x\to 2}\dfrac{x^3-8}{x^2+x-6} = 12/5$ Ans.

(iv) $\lim\limits_{x\to 1} \dfrac{x^3-2x^2+3x-1}{x^3-x}$

$= \lim\limits_{x\to 1} \dfrac{(x^3-1)-3x(x-1)}{x(x^2-1)}$

$= \lim\limits_{x\to 1} \dfrac{(x-1)(x^2+x+1)-3x(x-1)}{x(x+1)(x-1)}$

$= \lim\limits_{x\to 1} \dfrac{(x-1)\{x^2+x+1-3x\}}{x(x+1)(x-1)}$

$= \lim\limits_{x\to 1} \dfrac{x^2-2x+1}{x(x+1)} \Rightarrow \lim\limits_{x\to 1} \dfrac{(x-1)^2}{x(x+1)}$

$= \dfrac{(1-1)^2}{1(1+1)} = \dfrac{0}{2} = 0$

Thus $\lim\limits_{x\to 1} \dfrac{x^3-3x^2+3x-1}{x^3-x} = 0$ Ans.

(v) $\lim\limits_{x\to -1} \left(\dfrac{x^3+x^2}{x^2-1}\right)$

$= \lim\limits_{x\to -1} \dfrac{x^2(x+1)}{(x+1)(x-1)}$

$= \lim\limits_{x\to -1} \dfrac{x^2}{x-1}$

$= \dfrac{(-1)^2}{(-1)-1} = \dfrac{1}{-2}$

$\lim\limits_{x\to -1} \dfrac{x^3+x^2}{x^2-1} = -\frac{1}{2}$ Ans.

(vi) $\lim\limits_{x\to 4} \dfrac{2x^2-32}{x^3-4x^2}$

$= \lim\limits_{x\to 4} \dfrac{2(x^2-16)}{x^2(x-4)}$

$= \lim\limits_{x\to 4} \dfrac{2(x+4)(x-4)}{x^2(x-4)}$

$= \lim\limits_{x\to 4} \dfrac{2(x+4)}{x^2}$

$= \dfrac{2(4+4)}{(4)^2} = \dfrac{2(8)}{16} = \dfrac{16}{16}$

$\lim\limits_{x\to 4} \dfrac{2x^2-32}{x^3-4x^2} = 1$ Ans.

(vii) $\lim\limits_{x\to 2} \dfrac{\sqrt{x}-\sqrt{2}}{x-2}$

$= \lim\limits_{x\to 2} \dfrac{\sqrt{x}-\sqrt{2}}{(\sqrt{x})^2-(\sqrt{2})^2}$

$= \lim\limits_{x\to 2} \dfrac{\sqrt{x}-\sqrt{2}}{(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})}$

$= \lim\limits_{x\to 2} \dfrac{1}{\sqrt{x}+\sqrt{2}}$

$= \dfrac{1}{\sqrt{2}+\sqrt{2}} = \dfrac{1}{2\sqrt{2}}$

Thus $\lim\limits_{x\to 2} \dfrac{\sqrt{x}-\sqrt{2}}{x-2} = \dfrac{1}{2\sqrt{2}}$ Ans.

(viii) $\lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

$= \lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h} \times \dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$= \lim\limits_{h\to 0} \dfrac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})}$

$= \lim\limits_{h\to 0} \dfrac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$

$= \lim\limits_{h\to 0} \dfrac{h}{h(\sqrt{x+h}+\sqrt{x})}$

$= \lim\limits_{h\to 0} \dfrac{1}{\sqrt{x+h}+\sqrt{x}}$

$= \dfrac{1}{\sqrt{x+0}+\sqrt{x}} = \dfrac{1}{\sqrt{x}+\sqrt{x}}$

$= \dfrac{1}{2\sqrt{x}}$

Thus $\lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h} = \dfrac{1}{2\sqrt{x}}$ Ans.

(ix) $\lim_{x\to a} \dfrac{x^n - a^n}{x^m - a^m}$

Let $x = a+h$ and $h \to 0$

$= \lim_{h\to 0} \dfrac{(a+h)^n - a^n}{(a+h)^m - a^m}$

$= \lim_{h\to 0} \dfrac{a^n(1+\frac{h}{a})^n - a^n}{a^m(1+h/a)^m - a^m}$

Using binomial theorem

$= \lim_{h\to 0} \dfrac{a^n\{1+\frac{nh}{a} + \frac{n(n-1)}{2!}\frac{h^2}{a^2} + \cdots\} - a^n}{a^m\{1+\frac{mh}{a} + \frac{m(m-1)}{2!}\frac{h^2}{a^2} + \cdots\} - a^m}$

$= \lim_{h\to 0} \dfrac{a^n + nha^{n-1} + \frac{n(n-1)}{2!}h^2a^{n-2} + \cdots - a^n}{a^m + mha^{m-1} + \frac{m(m-1)}{2!}h^2a^{m-2} + \cdots + a^m}$

$= \lim_{h\to 0} \dfrac{h(na^{n-1} + \frac{n(n-1)}{2!}ha^{n-2} + \cdots)}{h(ma^{m-1} + \frac{m(m-1)}{2!}ha^{m-2} + \cdots)}$

$= \lim_{h\to 0} \dfrac{na^{n-1} + \frac{n(n-1)}{2!}ha^{n-2} + \cdots}{ma^{m-1} + \frac{m(m-1)}{2!}ha^{m-2} + \cdots}$

By Applying Limit

$= \dfrac{na^{n-1} + \frac{n(n-1)}{2!}(0)a^{n-2} + \cdots}{ma^{m-1} + \frac{m(m-1)}{2!}(0)a^{m-2} + \cdots}$

$= \dfrac{na^{n-1}+0}{ma^{m-1}+0} = \dfrac{na^{n-1}}{ma^{m-1}}$

Thus $\lim_{x\to a} \dfrac{x^n - a^n}{x^m - a^m} = \dfrac{na^{n-1}}{ma^{m-1}}$ Ans.

Q.3 Evaluate the followings:

(i) $\lim_{x\to 0} \dfrac{\sin 7x}{x}$

$= \lim_{x\to 0} \dfrac{\sin 7x}{7x} \cdot 7$

$= \lim_{x\to 0} \dfrac{\sin 7x}{7x} \cdot 7$

$\left(\because \lim_{x\to 0} \dfrac{\sin x}{x} = 1\right)$

$= 1 \cdot 7 = 7$

Thus $\lim_{x\to 0} \dfrac{\sin 7x}{x} = 7$ Ans.

(ii) $\lim_{x\to 0} \dfrac{\sin x^\circ}{x}$

Converting into radians.

$= \lim_{x\to 0} \dfrac{\sin \frac{x\pi}{180}}{x}$

$= \lim_{x\to 0} \dfrac{\sin \frac{\pi x}{180}}{\frac{\pi x}{180} \times \frac{180}{\pi}}$

$= \lim_{x\to 0} \dfrac{\sin \frac{\pi x}{180}}{\pi x/180} \cdot \dfrac{\pi}{180}$

$\left(\lim_{x\to 0} \dfrac{\sin x}{x} = 1\right)$

$= 1 \cdot \pi/180 = \pi/180$

Thus $\lim_{x\to 0} \dfrac{\sin x^\circ}{x} = \pi/180$ Ans.

(iii) $\lim_{\theta\to 0} \dfrac{1-\cos\theta}{\sin\theta}$

$\because \cos 2x = 1 - 2\sin^2 x$

$\sin 2x = 2\sin x\cos x$

$= \lim_{\theta\to 0} \dfrac{2\sin^2 \theta/2}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$= \lim_{\theta\to 0} \dfrac{\sin\theta/2}{\cos\theta/2}$

Applying Limit

$= \dfrac{\sin 0}{\cos 0}$ $\quad \because \sin 0^\circ = 0, \ \cos 0^\circ = 1$

$= \dfrac{0}{1} = 0$

Thus $\lim_{\theta\to 0} \dfrac{1-\cos\theta}{\sin\theta} = 0$

(iv) $\lim_{x\to \pi} \dfrac{\sin x}{\pi - x}$

Let $\theta = \pi - x \Rightarrow x = \pi - \theta$

As $x \to \pi$ $\quad \theta \to 0$

$\because \lim_{\theta\to 0} \dfrac{\sin(\pi-\theta)}{\theta}$

$\because \sin(\pi-\theta) = \sin\theta$

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$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$$

$$= 1$$

Thus $\lim_{x\to\pi}\frac{\sin x}{\pi-x} = 1$ Ans.

(v) $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$

$$= \lim_{x\to 0}\frac{\sin ax}{ax}\cdot ax\cdot\frac{1}{bx.\frac{\sin bx}{bx}}$$

$$= \frac{ax}{bx}\cdot\lim_{x\to 0}\frac{\sin ax}{ax}\cdot\lim_{x\to 0}\frac{1}{\frac{\sin bx}{bx}}$$

$$= \frac{a}{b}\cdot 1\cdot\frac{1}{1} \quad \left(\because \lim_{x\to 0}\frac{\sin x}{x}=1\right)$$

$$= a/b$$

Thus $\lim_{x\to 0}\frac{\sin ax}{\sin bx} = \frac{a}{b}$ Ans.

(vi) $\lim_{x\to 0}\frac{x}{\tan x}$

$$= \lim_{x\to 0}\frac{x}{\sin x}\cdot\cos x$$

$$= \lim_{x\to 0}\frac{1}{\left(\frac{\sin x}{x}\right)}\cdot\lim_{x\to 0}\cos x$$

$$= 1\cdot 1 = 1$$

Thus $\lim_{x\to 0}\frac{x}{\tan x} = 1$ Ans.

(vii) $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$

$$= \lim_{x\to 0}\frac{2\sin^2 x}{x^2}$$

$\because \cos 2\theta = 1-2\sin^2\theta$, $2\sin^2\theta = 1-\cos 2\theta$

$$= 2\cdot\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{x}$$

$$= 2\cdot 1\cdot 1 = 2$$

Thus

$\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = 2$ Ans.

(viii) $\lim_{x\to 0}\frac{1-\cos x}{\sin^2 x}$

$\because \sin^2 x+\cos^2 x = 1$, $\sin^2 x = 1-\cos^2 x$

$$= \lim_{x\to 0}\frac{1-\cos x}{1-\cos^2 x}$$

$$= \lim_{x\to 0}\frac{(1-\cos x)}{(1+\cos x)(1-\cos x)}$$

$$= \lim_{x\to 0}\frac{1}{1+\cos x}$$

Applying Limit

$$= \frac{1}{1+\cos 0} = \frac{1}{1+1} = \frac{1}{2}$$

Thus $\lim_{x\to 0}\frac{1-\cos x}{\sin^2 x} = \frac{1}{2}$ Ans.

(ix) $\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta}$

$$= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\sin\theta$$

$$= 1\cdot 0 \quad \because \sin 0 = 0$$

$$= 0$$

Thus $\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta} = 0$ Ans.

(x) $\lim_{x\to 0}\frac{\sec x-\cos x}{x}$

$$= \lim_{x\to 0}\frac{\frac{1}{\cos x}-\cos x}{x}$$

$$= \lim_{x\to 0}\frac{1-\cos^2 x}{x\cos x}$$

$$= \lim_{x\to 0}\frac{\sin^2 x}{x\cos x}$$

$$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{\cos x}$$

$$= 1\cdot\frac{0}{1} = 0$$

Thus $\lim_{x\to 0}\frac{\sec x-\cos x}{x} = 0$ Ans.

(xii) $\lim_{\theta\to 0} \dfrac{1-\cos p\theta}{1-\cos q\theta}$ $\quad 1-\cos 2\theta = 2\sin^2\theta$

$= \lim_{\theta\to 0} \dfrac{2\sin^2\frac{p}{2}\theta}{2\sin^2\frac{q}{2}\theta}$

$= \lim_{\theta\to 0} \dfrac{\sin^2\frac{p}{2}\theta}{(\frac{p\theta}{2})^2}\cdot(\frac{p\theta}{2})^2\cdot\dfrac{1}{\frac{\sin^2\frac{q\theta}{2}}{(\frac{q\theta}{2})^2}\cdot(\frac{q\theta}{2})^2}$

$= \lim_{\theta\to 0}\left(\dfrac{\sin\frac{p\theta}{2}}{\frac{p\theta}{2}}\right)^2\cdot\left(\dfrac{p\theta\, 2}{2\cdot q\theta}\right)^2\cdot\dfrac{1}{\left(\frac{\sin\frac{q\theta}{2}}{\frac{q\theta}{2}}\right)^2}$

$= \left\{\lim_{\theta\to 0}\dfrac{\sin\frac{p\theta}{2}}{p\theta/2}\right\}^2\cdot\left(\dfrac{p}{q}\right)^2\cdot\dfrac{1}{\left\{\lim_{\theta\to 0}\frac{\sin\frac{q\theta}{2}}{\frac{q\theta}{2}}\right\}^2}$

$= 1\cdot\left(\dfrac{p}{q}\right)^2\cdot\dfrac{1}{1}$

$= \dfrac{p^2}{q^2}$ Ans.

Thus $\lim_{\theta\to 0}\dfrac{1-\cos p\theta}{1-\cos q\theta} = \dfrac{p^2}{q^2}$ Ans.

(xiii) $\lim_{\theta\to 0}\dfrac{\tan\theta-\sin\theta}{\sin^3\theta}$

$= \lim_{\theta\to 0}\dfrac{\left(\frac{\sin\theta}{\cos\theta}-\sin\theta\right)}{\sin\theta\,\sin^2\theta}$

$= \lim_{\theta\to 0}\dfrac{\sin\theta\left(\frac{1-\cos\theta}{\cos\theta}\right)}{\sin\theta(1-\cos^2\theta)}$

$= \lim_{\theta\to 0}\dfrac{\cancel{\sin\theta}\,\cancel{(1-\cos\theta)}}{\cancel{\sin\theta}\cos\theta\,\cancel{(1-\cos\theta)}(1+\cos\theta)}$

$= \lim_{\theta\to 0}\dfrac{1}{\cos\theta(1+\cos\theta)}$

$= \dfrac{1}{1.(1+1)} = \dfrac{1}{2}$

Thus $\lim_{\theta\to 0}\dfrac{\tan\theta-\sin\theta}{\sin^3\theta} = \frac{1}{2}$ Ans.

Q.4 Express each in terms of e:

(i) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}$

$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^2$

$= \left(\lim_{n\to\infty}\left(1+1/n\right)^n\right)^2$

$= (e)^2 = e^2$

Thus $\lim_{n\to\infty}(1+1/n)^{2n} = e^2$ Ans.

$\because \left\{\begin{array}{l}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \\ \text{or } \lim_{n\to 0}(1+n)^{1/n} = e\end{array}\right.$

(ii) $\lim_{n\to\infty}(1+1/n)^{n/2}$

$= \lim_{n\to\infty}\left[\left(1+1/n\right)^n\right]^{1/2}$

$= \left[\lim_{n\to\infty}(1+1/n)^n\right]^{1/2}$

$= (e)^{1/2} = \sqrt{e}$

Thus $\lim_{n\to\infty}(1+1/n)^{n/2} = \sqrt{e}$ Ans.

(iii) $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$

$= \lim_{n\to\infty}\left[\left(1+(-1/n)\right)^{-n}\right]^{-1}$

$= \left[\lim_{n\to\infty}\left(1+(-1/n)\right)^{-n}\right]^{-1}$

$= (e)^{-1} = e^{-1} = 1/e$

Thus $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = 1/e$ Ans.

(iv) $\lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^n$

$= \lim_{n\to\infty}\left[\left(1+\frac{1}{3n}\right)^{3n}\right]^{1/3}$

$= \left[\lim_{n\to\infty}\left(1+1/3n\right)^{3n}\right]^{1/3} = (e)^{1/3} = e^{1/3}$

Thus $\lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^n = e^{1/3}$ Ans.

(v) $\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^n$

$= \lim_{n\to\infty}\left[\left(1+\frac{4}{n}\right)^{n/4}\right]^4$

$= \left[\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^{n/4}\right]^4$

$= (e)^4 = e^4$

Thus

$\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^n = e^4$ Ans.

(vi) $\lim_{x\to 0}(1+3x)^{2/x}$

$= \lim_{x\to 0}\left[(1+3x)^{\frac{1}{3x}}\right]^{3\cdot 2}$

$= \left[\lim_{x\to 0}(1+3x)^{1/3x}\right]^6$

$= (e)^6 = e^6$

$\because \lim_{x\to 0}(1+x)^{1/x} = e$

Thus $\lim_{x\to 0}(1+3x)^{2/x} = e^6$ Ans.

(vii) $\lim_{x\to 0}(1+2x^2)^{1/x^2}$

$= \lim_{x\to 0}\left[(1+2x^2)^{\frac{1}{2x^2}}\right]^2$

$= \left[\lim_{x\to 0}(1+2x^2)^{1/2x^2}\right]^2$

$= (e)^2 = e^2$

Thus $\lim_{x\to 0}(1+2x^2)^{1/x^2} = e^2$ Ans.

(viii) $\lim_{h\to 0}(1-2h)^{1/h}$

$= \lim_{h\to 0}\left[\left[1+(-2h)\right]^{\frac{1}{-2h}}\right]^{-2}$

$= \left\{\lim_{h\to 0}\left(1+(-2h)\right)^{\frac{1}{-2h}}\right\}^{-2}$

$= (e)^{-2} = e^{-2} = 1/e^2$

Thus $\lim_{h\to 0}(1-2h)^{1/h} = 1/e^2$ Ans.

(ix) $\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^x$

$= \lim_{x\to\infty}\frac{1}{\left[\frac{x}{1+x}\right]^{-x}}$ $\quad\left(\because \frac{1}{x^{-n}} = x^n\right)$

$= \lim_{x\to\infty}\frac{1}{\left(\frac{1+x}{x}\right)^x}$

$= \frac{1}{\lim_{x\to\infty}\left(\frac{1}{x}+\frac{x}{x}\right)^x}$

$= \frac{1}{\lim_{x\to\infty}(1+1/x)^x}$

$= \frac{1}{e}$ Ans.

Thus $\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^x = 1/e$ Ans.

(x) $\lim_{x\to 0}\frac{e^{1/x}-1}{e^{1/x}+1}$ $\quad x<0$

Let $y = 1/x$ as $x\to 0$ $y\to -\infty$ $\because x<0$

$= \lim_{y\to-\infty}\frac{e^y-1}{e^y+1}$

as $e^y \to 0$ as $y\to-\infty$

$= \frac{0-1}{0+1} = -1$

Thus $\lim_{x\to 0}\frac{e^{1/x}-1}{e^{1/x}+1} = -1$ Ans for $x<0$

(xi) $\lim_{x\to 0}\frac{e^{1/x}-1}{e^{1/x}+1}$ for $x>0$

Let $y = 1/x$ as $x\to 0$ $y\to\infty$

$= \lim_{y\to\infty}\frac{e^y-1}{e^y+1} = \lim_{y\to\infty}\frac{1-1/e^y}{1+1/e^y}$

(Dividing N and D by e^y)

$= \lim_{y\to\infty}\frac{1-1/e^y}{1+1/e^y}$ $\quad$ as $y\to\infty$ $\frac{1}{e^y}\to 0$

$= \frac{1-0}{1+0} = 1$

Thus $\lim_{x\to 0}\frac{e^{1/x}-1}{e^{1/x}+1} = 1$ Ans. $\forall x>0$