(vi) $f(x) = \dfrac{x^3 - x}{x^2+1}$

$f(-x) = \dfrac{(-x)^3 - (-x)}{(-x)^2+1}$

$f(-x) = \dfrac{-x^3 + x}{x^2+1}$

$f(-x) = -\left[\dfrac{x^3 - x}{x^2+1}\right]$

$f(-x) = -f(x)$

Thus $f(x)$ is Odd function.

Composite Function :-

"Let f and g are two functions then $f\circ g$ and $g\circ f$ are called Composite function of f and g."

$f\circ g = f[g(x)]$ {f circle g}

$g\circ f = g[f(x)]$ {g circle f}

* $f\circ g \neq g\circ f$ and $D_{f\circ g} = D_f$, $R_{f\circ g} = R_g$.

Inverse of a function :-

Let f be a function from X to Y then inverse function of f is denoted as f^{-1} and is defined as a function from Y to X.

If $y = f(x)$ then $f^{-1}(y) = x$ is called inverse function

* For finding inverse function, function must be one-one and onto.

* $f(x) \circ f^{-1}(x) = x$

* $f(y) \circ f^{-1}(y) = y$

Exercise: 1.2

Q.1 Find the followings:

(i) $f(x) = 2x+1$, $g(x) = \dfrac{3}{x-1}$

(a) $f\circ g(x) = ?$

$f\circ g(x) = f[g(x)]$

$f\circ g(x) = 2\left(\dfrac{3}{x-1}\right) + 1$

$f\circ g(x) = \dfrac{6 + x - 1}{x-1} = \dfrac{5+x}{x-1}$

$f\circ g(x) = \dfrac{x+5}{x-1}$ Ans.

(b) $g\circ f(x) = ?$

$g\circ f(x) = g[f(x)]$

$g\circ f(x) = \dfrac{3}{(2x+1)-1} = \dfrac{3}{2x+1-1}$

$g\circ f(x) = \dfrac{3}{2x}$ Ans.

(c) $f\circ f(x) = ?$

$f\circ f(x) = f[f(x)]$

$f\circ f(x) = 2[2x+1] + 1$

$f\circ f(x) = 4x+2+1 = 4x+3$ Ans.

(d) $g\circ g(x) = ?$

$g\circ g(x) = g[g(x)]$

$g\circ g(x) = \dfrac{3}{\left(\frac{3}{x-1}\right)-1} = \dfrac{3(x-1)}{3-1(x-1)}$

$g\circ g(x) = \dfrac{3(x-1)}{3-x+1} = \dfrac{3(x-1)}{4-x}$ Ans.

(ii) $f(x) = \sqrt{x+1}$, $g(x) = 1/x^2$

(a) $f\circ g(x) = ?$

$f\circ g(x) = \sqrt{\left(\frac{1}{x^2}\right)+1} = \dfrac{\sqrt{1+x^2}}{x}$ Ans.

(b) $g\circ f(x) = ?$

$$g\circ f(x) = \frac{1}{(\sqrt{x+1})^2}$$

$$g\circ f(x) = \frac{1}{x+1} \quad \text{Ans.}$$

(c) $f\circ f(x) = ?$

$$f\circ f(x) = \sqrt{(\sqrt{x+1})+1} \quad \text{Ans.}$$

(d) $g\circ g(x) = ?$

$$g\circ g(x) = \frac{1}{(1/x^2)^2}$$

$$g\circ g(x) = \frac{x^4}{1} = x^4 \quad \text{Ans.}$$

(iii) $f(x) = \frac{1}{\sqrt{x-1}}$, $g(x) = (x^2+1)^2$

(a) $f\circ g(x) = ?$

$$f\circ g(x) = \frac{1}{\sqrt{(x^2+1)^2-1}} = \frac{1}{\sqrt{x^4+1+2x^2-1}}$$

$$f\circ g(x) = \frac{1}{\sqrt{x^4+2x^2}} = \frac{1}{x\sqrt{x^2+2}} \quad \text{Ans.}$$

(b) $g\circ f(x) = ?$

$$g\circ f(x) = \left[\left(\frac{1}{\sqrt{x-1}}\right)^2+1\right]^2 = \left[\frac{1}{x-1}+1\right]^2$$

$$g\circ f(x) = \left[\frac{1+x-1}{x-1}\right]^2 = \left(\frac{x}{x-1}\right)^2 \quad \text{Ans.}$$

(c) $f\circ f(x) = ?$

$$f\circ f(x) = \frac{1}{\sqrt{\left(\frac{1}{\sqrt{x-1}}\right)-1}}$$

$$f\circ f(x) = \frac{1}{\sqrt{\frac{1-\sqrt{x-1}}{\sqrt{x-1}}}}$$

$$f\circ f(x) = \sqrt{\frac{\sqrt{x-1}}{1-\sqrt{x-1}}} \quad \text{Ans.}$$

(d) $g\circ g(x) = ?$

$$g\circ g(x) = \left[\left((x^2+1)^2\right)^2+1\right]^2$$

$$g\circ g(x) = \left((x^2+1)^4+1\right)^2 \quad \text{Ans.}$$

(iv) $f(x) = 3x^4-2x^2$, $g(x) = \frac{2}{\sqrt{x}}$

(a) $f\circ g(x) = ?$

$$f\circ g(x) = 3\left(\frac{2}{\sqrt{x}}\right)^4 - 2\left(\frac{2}{\sqrt{x}}\right)^2$$

$$f\circ g(x) = 3\left(\frac{16}{x^2}\right) - 2\left(\frac{4}{x}\right) = \frac{48}{x^2} - \frac{8}{x}$$

$$f\circ g(x) = \frac{48-8x}{x^2} \quad \text{Ans.}$$

(b) $g\circ f(x) = ?$

$$g\circ f(x) = \frac{2}{\sqrt{3x^4-2x^2}}$$

$$g\circ f(x) = \frac{2}{x\sqrt{3x^2-2}} \quad \text{Ans.}$$

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779

(c) $f\circ f(x) = ?$

$$f\circ f(x) = 3(3x^4-2x^2)^4 - 2(3x^4-2x^2)^2 \quad \text{Ans.}$$

(d) $g\circ g(x) = ?$

$$g\circ g(x) = \frac{2}{\sqrt{2/\sqrt{x}}} = \frac{2\sqrt{\sqrt{x}}}{\sqrt{2}}$$

$$g\circ g(x) = \sqrt{2\sqrt{x}} \quad \text{Ans.}$$

Q.2 Find the followings:

(i) $f(x) = -2x+8$

(a) $f^{-1}(x) = ?$

Let $y = f(x) = -2x+8$

$$y = -2x+8 \Rightarrow 2x = 8-y$$

$$x = \frac{8-y}{2} \qquad \because f(x) = y,\; x = f^{-1}(y)$$

$f^{-1}(y) = \frac{8-y}{2}$

Replacing y by x, we have

$f^{-1}(x) = \frac{8-x}{2}$ Ans.

(b) $f^{-1}(-1) = ?$

$f^{-1}(-1) = \frac{8-(-1)}{2} = \frac{8+1}{2} = 9/2$

$f^{-1}(-1) = 9/2$ Ans.

verify $f[f^{-1}(x)] = f^{-1}[f(x)] = x$

$f[f^{-1}(x)] = -2\left(\frac{8-x}{2}\right) + 8$

$f(f^{-1}(x)) = -8 + x + 8 = x$ —— (i)

$\Rightarrow f^{-1}(f(x)] = \frac{8-(-2x+8)}{2}$

$f^{-1}(f(x)] = \frac{8+2x-8}{2} = \frac{2x}{2}$

$f^{-1}[f(x)] = x$ —— (ii)

From (i) and (ii)

$f[f^{-1}(x)] = f^{-1}[f(x)] = x$ (Proved).

(iii) $f(x) = 3x^3 + 7$

(a) $f^{-1}(x) = ?$

Let $y = f(x) = 3x^3 + 7$

$y = 3x^3 + 7 \Rightarrow 3x^3 = y - 7$

$x^3 = \frac{y-7}{3} \Rightarrow x = \left[\frac{y-7}{3}\right]^{1/3}$

$f^{-1}(y) = \left[\frac{y-7}{3}\right]^{1/3}$ $\because y = f(x)$, $f^{-1}(y) = x$

Replacing y by x, we have

$f^{-1}(x) = \left[\frac{x-7}{3}\right]^{1/3}$ Ans.

(b) $f^{-1}(-1) = ?$

$f^{-1}(-1) = \left(\frac{-1-7}{3}\right)^{1/3} \Rightarrow f^{-1}(-1) = \left(-\frac{8}{3}\right)^{1/3}$

$f^{-1}(-1) = \left(-8/3\right)^{1/3}$ Ans.

Verify $f[f^{-1}(x)] = f^{-1}[f(x)] = x$

$f[f^{-1}(x)] = 3\left(\frac{x-7}{3}\right)^{3/3} + 7$

$f[f^{-1}(x)] = 3\left(\frac{x-7}{3}\right) + 7 = x - 7 + 7$

$f[f^{-1}(x)] = x$ —— (i)

$\Rightarrow f^{-1}[f(x)] = \left(\frac{(3x^3+7)-7}{3}\right)^{1/3}$

$f^{-1}[f(x)] = \left(\frac{3x^3}{3}\right)^{1/3} = (x^3)^{1/3}$

$f^{-1}(f(x)] = x$ —— (ii)

From (i) and (ii), we have

$f[f^{-1}(x)] = f^{-1}[f(x)] = x$ (Proved).

(iii) $f(x) = (-x+9)^3$

(a) $f^{-1}(x) = ?$

Let $y = f(x) = (-x+9)^3$

$y = (-x+9)^3 \Rightarrow y^{1/3} = -x + 9$

$x = 9 - y^{1/3}$ $\because f(x) = y$

$f^{-1}(y) = 9 - y^{1/3}$ $f^{-1}(y) = x$

Replacing y by x, we have

$f^{-1}(x) = 9 - x^{1/3}$ Ans.

(b) $f^{-1}(-1) = ?$

$f^{-1}(-1) = 9 - (-1)^{1/3}$ Ans.

Verify $f[f^{-1}(x)] = f^{-1}(f(x)] = x$

$f[f^{-1}(x)] = [-(9 - x^{1/3}) + 9]^3 = (-9 + x^{1/3} + 9)^3$

$f[f^{-1}(x)] = (x^{1/3})^3 = x$ —— (i)

$f^{-1}[f(x)] = 9 - [(-x+9)^3]^{1/3}$

$f^{-1}(f(x)] = 9 + x - 9 = x$ —— (ii)

From (i) and (ii)

$f[f^{-1}(x)] = f^{-1}(f(x)] = x$ (Proved).

(iv) $f(x) = \dfrac{2x+1}{x-1}$

(a) $f^{-1}(x) = ?$

Let $y = f(x) = \dfrac{2x+1}{x-1}$

$y = \dfrac{2x+1}{x-1} \Rightarrow y(x-1) = 2x+1$

$xy - y = 2x+1$

$xy - 2x = 1 + y$

$x(y-2) = 1+y \Rightarrow x = \dfrac{1+y}{y-2}$

$f^{-1}(y) = \dfrac{1+y}{y-2}$ $\because f(x) = y$, $f^{-1}(y) = x$

Replacing y by x

$f^{-1}(x) = \dfrac{1+x}{x-2}$ Ans.

(b) $f^{-1}(-1) = ?$

$f^{-1}(-1) = \dfrac{1+(-1)}{(-1)-2} = \dfrac{1-1}{-1-2} = \dfrac{0}{-3}$

$f^{-1}(-1) = 0$ Ans.

Verify $f[f^{-1}(x)] = f^{-1}[f(x)] = x$

$f[f^{-1}(x)] = \dfrac{2\left[\frac{1+x}{x-2}\right]+1}{\left[\frac{1+x}{x-2}\right]-1}$

$f[f^{-1}(x)] = \dfrac{2(1+x)+(x-2)}{(1+x)-(x-2)} \times \dfrac{(x-2)}{(x-2)}$

$f[f^{-1}(x)] = \dfrac{2+2x+x-2}{1+x-x+2} = \dfrac{3x}{3}$

$f[f^{-1}(x)] = x$ —— (i)

$f^{-1}[f(x)] = \dfrac{1+\left(\frac{2x+1}{x-1}\right)}{\left(\frac{2x+1}{x-1}\right)-2}$

$f^{-1}[f(x)] = \dfrac{x-1+2x+1}{2x+1-2(x-1)} \times \dfrac{x-1}{x-1}$

$f^{-1}(f(x)) = \dfrac{3x}{2x+1-2x+2} = \dfrac{3x}{3} = x$

$f^{-1}(f(x)) = x$ —— (ii)

From (i) and (ii)

$f[f^{-1}(x)] = f^{-1}[f(x)] = x$ (Proved)

Q.3 Without find inverse, give domain and range of f^{-1}?

$*$ $Dom(f) = Ran(f^{-1})$
$*$ $Ran(f) = Dom(f^{-1})$

(i) $f(x) = \sqrt{x+2}$

f(x) becomes imaginary when $x < -2$, thus

$D_f = [-2, \infty[$

$R_f = [0, \infty[$

Thus $D_{f^{-1}} = [0, \infty[$

$R_{f^{-1}} = [-2, \infty[$

(ii) $f(x) = \dfrac{x-1}{x-4}$ $x \neq 4$

Clearly $f(x)$ becomes ∞ for $x=4$ otherwise $f(x)$ is real valued so

$Dom(f) = \mathbb{R} - \{4\}$

$Ran(f) = \mathbb{R} - \{3\}$

Thus $Dom(f^{-1}) = \mathbb{R} - \{3\}$

$Ran(f^{-1}) = \mathbb{R} - \{4\}$

(iii) $f(x) = \dfrac{1}{x+3}$ $x \neq -3$

Clearly $f(x)$ becomes ∞ for $x = -3$

Thus $Dom(f) = \mathbb{R} - \{-3\}$

$Ran(f) = \mathbb{R} - \{1\}$

Thus $Dom(f^{-1}) = \mathbb{R} - \{1\}$

$Ran(f^{-1}) = \mathbb{R} - \{-3\}$

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779