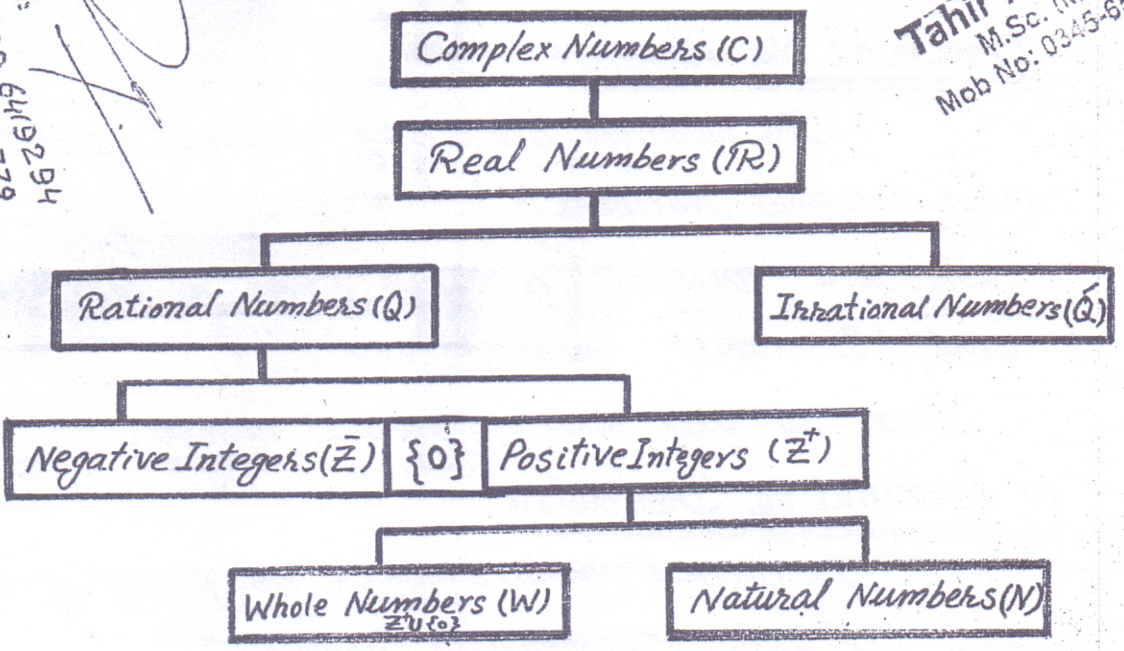
Number System



The Set of Natural Numbers (N)

$$N = \{1, 2, 3, 4, 5, \ldots\}$$

The Set of Whole Numbers (W)

$$W = \{0, 1, 2, 3, 4, 5, \ldots\}$$

The Set of Integers (Z)

$$Z = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \ldots\}$$

* The letter Z for integers is the abbrevation of Germen word "Zahlem" means counting numbers.

The Set of Rational numbers (Q)

$$Q = \{x \mid x \in Q \wedge x = p/q,\ p, q \in Z,\ q \neq 0\}$$ e.g $\frac{2}{3}, 0, \frac{1}{7}$ etc

The Set of Irrational numbers (Q')

$$Q' = \{x \mid x \notin Q\}$$ e.g. $\pi, e, \sqrt{2}, \sqrt{3}$ etc.

The Set of Real numbers ($\mathbb{R}$)

The set of all rational and irrational numbers is called set of real numbers $\mathbb{R} = Q \cup Q'$

Decimal Numbers:-

"The numbers containing decimal symbol (·) are called Decimal Numbers."

The word 'Decimal' is taken from a Germen (Latin) word "Decimus" means tenth.

There are three types of decimal numbers.

(i) Terminating Decimals:-

"The decimal numbers which are finite of decimal places are called Terminating Decimals."

For example 4.5 , 12.397 etc

* These numbers are rational (Q).

(ii) Recurring Decimals:-

"The decimal numbers which are not finite of decimal places but two or more than two digits repeats again and again are called Recurring Decimals."

For example 3.44334433.... , 4.333333, 2.123123123.... etc

* These are also rational numbers (Q).

(iii) Non-Recurring, Non-Terminating Decimals:-

"The decimal numbers which as non-terminating and non-recurring are called Non-terminating, Non-recurring decimals."

For example $\sqrt{n}$, $\sqrt{5}$, π, 2.1314151617.......

where n is odd having no complete Square.

* These are irrational numbers (Q').

Theorem:-

The square root of a positive integer, which is not a perfect square, is irrational.

Proof:-

Let n be a positive integer which is not a perfect square, then we are to show that $\sqrt{n}$ is an irrational number. For this we suppose on contrary (الٹ) that $\sqrt{n}$ is a rational number then there exist relatively prime integers p and q such that

$$\sqrt{n} = \frac{p}{q} \Rightarrow q \neq 0$$

Squaring on both sides

$$n = \frac{p^2}{q^2} \Rightarrow nq = \frac{p^2}{q}$$

Since p and q are relatively prime so q does not devide p and p^2. This shows that $\frac{p^2}{q}$ is not an integer but $qn = \frac{p^2}{q}$ shows that qn also not an integer. which is a contradiction because qn being the product of two integers is an integer.

Hence $\sqrt{n}$ is an irrational number.

* Square root of prime is irrational.

* The two integers are said to be relatively prime if their (H.C.F) or (G.C.D) is 1.

* On the justification of above theorem, we can prove that $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \ldots$ are irrational.

Unary Operation:-

"The mathematical rule that provides another number when applied upon any number is called Unary Operation."

For example $\sqrt{x}$ (Square root) and $(x)^2, (x)^3, \ldots$ are unary operations.

Binary Operation:-

"The mathematical rule that provides a third number when applied upon any two numbers is called Binary Operation."

* Addition and Multiplication are two important binary operations and are denoted respectively $(+)$ and $(\cdot$ or $\times)$.

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779

Properties of Real Number ($\mathbb{R}$):-

(a) W.R.T. Addition:-

(i) Closure Property w.r.t. addition:-

$\forall\ a, b \in \mathbb{R}$ ($\mathbb{R}$ are closed under addition).

$a+b \in \mathbb{R}$

(ii) Commutative Property w.r.t addition:-

$\forall\ a, b \in \mathbb{R}$

$a+b = b+a$

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779

(iii) Associative Property w.r.t. addition:-

$\forall\ a, b, c \in \mathbb{R}$

$a+(b+c) = (a+b)+c$

(iv) Additive Identity Property:-

$\forall\ a \in \mathbb{R}$, $\exists\ 0 \in \mathbb{R}$ such that

$a+0 = 0+a = a$

0 is called identity element of addition.

(v) Additive inverse Property:-

$\forall\ a \in \mathbb{R}$, $\exists\ -a \in \mathbb{R}$ such that

$a+(-a) = (-a)+(a) = 0$

$-a$ is called additive inverse of a.

(b) W.R.T. Multiplication:-

(i) Closure Property w.r.t Multiplication:-

$\forall\ a, b \in \mathbb{R}$ ($\mathbb{R}$ is closed under multiplication)

$ab \in \mathbb{R}$

(ii) Commutative Property w.r.t. $\times$.

$\forall\ a, b \in \mathbb{R}$

$a \cdot b = b \cdot a$

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(iii) Associative Property w.r.t $\times$.

$\forall\ a, b, c \in \mathbb{R}$

$a \cdot (b \cdot c) = (a \cdot b) \cdot c$

(iv) Multiplicative Identity Property:-

$\forall\ a \in \mathbb{R}$, $\exists\ 1 \in \mathbb{R}$ such that

$1 \cdot a = a \cdot 1 = a$

1 is called identity element for $\times$.

(v) Multiplicative Inverse Property:-

$\forall\ a \in \mathbb{R}$, $\exists\ \frac{1}{a} \in \mathbb{R}$ such that

$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$

$\frac{1}{a}$ is called multiplicative inverse of a and also denoted as a^{-1} or a^{*}.

(c) Muliplication-Addition Property:-

Distributive property of $\times$ w.r.t $+, -$.

$\forall\ a, b, c \in \mathbb{R}$

$a \cdot (b \pm c) = a \cdot b \pm a \cdot c$

is called Left distributive Property

$\forall\ a, b, c \in \mathbb{R}$

$(a \pm b) \cdot c = a \cdot c \pm b \cdot c$

is called Right distributive Property.

* W.R.T stands for "with respect to"
* $\forall$ stands for "For all".
* $\exists$ stands for "There exist".
* "$\times$" for Multiplication.
* "$+$" for addition.
* "$-$" stands for subtraction.

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779

Field:-

"Any set of numbers which satisfy all the 11 properties of real numbers is called Field."

Properties of Equality:-

"The properties of real numbers containg sign of equality (=) are called Equality Properties."

(i) Reflexive Property:-

$\forall a \in \mathbb{R}, \quad a = a$

(ii) Symmetric Property:-

$\forall a, b \in \mathbb{R}, \quad a = b \Longrightarrow b = a$

(iii) Transitive Property:-

$\forall a, b, c \in \mathbb{R}, \quad a = b \wedge b = c \Longrightarrow a = c$

(iv) Additive Property:-

$\forall a, b, c \in \mathbb{R}$

$a = b \Longrightarrow a + c = b + c$ (Right Additive)

$a = b \Longrightarrow c + a = c + b$ (Left Additive P.)

(v) Multiplicative Property:-

$\forall a, b, c \in \mathbb{R}$,

$a = b \Longrightarrow ac = bc$ (Right Multiplicative)

$a = b \Longrightarrow ca = cb$ (Left Multiplicative)

(vi) Cancellation property w.r.t. Addition:-

$\forall a, b, c \in \mathbb{R}$

$a + c = b + c \Longrightarrow a = b$ (Right Cancellation)

$c + a = c + b \Longrightarrow a = b$ (Left Cancellation P.)

(vii) Cancellation Property w.r.t. $\times$.

$\forall a, b, c \in \mathbb{R}$

$ac = bc \Longrightarrow a = b$ (Right Cancellation P.)

$ca = cb \Longrightarrow a = b$ (Left Cancellation Pro.)

Order Properties:-

"The properties containing inequality signs ($<, >$) are called inequality or Order Properties."

(i) Trichotomy (ثلاثی) Property:-

$\forall a, b \in \mathbb{R}$

Either $a > b$ or $a = b$ or $a < b$

(ii) Transitive Property:-

$\forall a, b, c \in \mathbb{R}$

(i) $a < b \wedge b < c \Longrightarrow a < c$

(ii) $a > b \wedge b > c \Longrightarrow a > c$

(iii) Additive Property:-

$\forall a, b, c \in \mathbb{R}$,

(i) $a > b \Longrightarrow a + c > b + c$

(ii) $a < b \Longrightarrow a + c < b + c$

(iv) Multiplicative Property:-

$\forall a, b, c \in \mathbb{R}$

(i) $a > b \Longrightarrow ac > bc$ for $c > 0$

$a < b \Longrightarrow ac < bc$

(ii) $a > b \Longrightarrow ac < bc$ for $c < 0$

$a < b \Longrightarrow ac > bc$

(V) Multiplicative Inverse inequality:-

$\forall\ a, b \in \mathbb{R}$

(i) $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}$

(ii) $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$

*** Additive inverse of an additive inverse** of number a is the number itself.

ie. $-(-a) = a$

* Multiplicative inverse of a multiplicative inverse number is the number itself.

ie. $(a^{-1})^{-1} = a$

Exercise:1.1

Q.1 Which of the following set is closed under addition and multiplication?

(i) $\{0\}$

Let $A = \{0\}$

$0 \in A$	$0 \in A$
$0+0 = 0 \in A$	$0 \cdot 0 = 0 \in A$

$\{0\}$ is closed under addition as well as multiplication.

(ii) $\{1\}$

Let $A = \{1\}$

$1 \in A$	$1 \in A$
$1+1 = 2 \notin A$	$1 \cdot 1 = 1 \in A$

$\{1\}$ is closed under multiplication but not closed under addition.

(iii) $\{0, -1\}$

Let $A = \{0, -1\}$

$0, -1 \in A$	$0, -1 \in A$
$0+0 = 0 \in A$	$0 \cdot 0 = 0 \in A$
$0+(-1) = -1 \in A$	$0 \cdot (-1) = 0 \in A$
$-1+0 = -1 \in A$	$(-1) \cdot 0 = 0 \in A$
$-1+(-1) = -2 \notin A$	$(-1) \cdot (-1) = 1 \notin A$

$\{0, -1\}$ is neither closed w.r.t. addition nor closed w.r.t. Multiplication.

(iv) $\{1, -1\}$

Let $A = \{1, -1\}$

$1, -1 \in A$	$1, -1 \in A$
$1+(-1) = 0 \notin A$	$1 \cdot 1 = 1 \in A$
$1+1 = 2 \notin A$	$-1 \cdot 1 = -1 \in A$
$-1+(-1) = -2 \notin A$	$1 \cdot (-1) = -1 \in A$
$-1+1 = 0 \notin A$	$(-1) \cdot (-1) = 1 \in A$

$\{1, -1\}$ is closed under multiplication but not closed under addition.

Q.2 Name the property that is used:

(i) $4+9=9+4$

Commutative Property w.r.t "+"

(ii) $(a+1)+\frac{3}{4}=a+(1+\frac{3}{4})$

Associative Property w.r.t "+"

(iii) $(\sqrt{3}+\sqrt{5})+\sqrt{7}=\sqrt{3}+(\sqrt{5}+\sqrt{7})$

Associative Property w.r.t "+"

(iv) $100+0=100$

Additive Identity Property

(v) $1000\times 1=1000$

Multiplicative identity Property.

(vi) $4.1+(-4.1)=0$

Additive inverse Property.

(vii) $a-a=0 \Rightarrow a+(-a)=0$

Additive inverse property.

(viii) $\sqrt{2}\times\sqrt{5}=\sqrt{5}\times\sqrt{2}$

Commutative Property w.r.t "X"

(ix) $a(b-c)=ab-ac$

Left distribute Property of "X" w.r.t "−"

(x) $(x-y)z=xz-yz$

Right distributive property of "X" w.r.t "−"

(xi) $4\times(5\times 8)=(4\times 5)\times 8$

Associative Property w.r.t "X"

(xii) $a(b+c-d)=ab+ac-ad$

Left distributive Property of "X" w.r.t "+" and "−"

Q.3: Name the Properties?

(i) $-3<-2 \Rightarrow 0<1$

$\Rightarrow -3+3<-2+3 \Rightarrow 0<1$

Additive Property.

(ii) $-5<-4 \Rightarrow 20>16$

$-5\times -4 > -4\times -4 \Rightarrow 20>16$

Multiplicative Property with $-4<0$

(iii) $1>-1 \Rightarrow -3>-5$

$1+(-4)>-1+(-4) \Rightarrow -3>-5$

Additive Property.

(iv) $a<0 \Rightarrow -a>0$

$a\times -1 > 0\times -1 \Rightarrow -a>0$

Multiplicative Property by $-1<0$

(v) $a>b \Rightarrow \frac{1}{a}<\frac{1}{b}$

Multiplicative inverse property.

(vi) $a>b \Rightarrow -a<-b$

$a\times(-1)<b\times(-1) \Rightarrow -a<-b$

Multiplicative Property by $-1<0$

Q.4 **Prove that:**

(i) $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$

L.H.S. $=\frac{a}{c}+\frac{b}{c}$

By Multiplicative identity Property

$=\left(\frac{a}{c}+\frac{b}{c}\right)\cdot 1$

$=\left(\frac{a}{c}+\frac{b}{c}\right)\left(c\cdot\frac{1}{c}\right)$ $\quad (\because a\cdot\frac{1}{a}=\frac{1}{a}\cdot a=1)$

$=\left(\frac{a}{c}\cdot c+\frac{b}{c}\cdot c\right)\frac{1}{c}$

By distributive Property.

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779

$= (a \cdot 1 + b \cdot 1) \cdot \frac{1}{c}$ $(\because a \cdot \frac{1}{a} = 1)$

$= (a + b) \frac{1}{c}$ $(a \cdot 1 = a)$

$= \left(\frac{a+b}{c}\right)$ (Proved).

(ii) $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

$LHS \cdot \left(\frac{a}{b} + \frac{c}{d}\right)$

$= \left(\frac{a}{b} + \frac{c}{d}\right) \cdot 1$ $(\because a \cdot 1 = a)$

$(\because a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1)$

$= \left(\frac{a}{b} + \frac{c}{d}\right) \cdot bd \cdot \frac{1}{bd}$

$= \frac{\left(\frac{a}{b} \cdot bd + \frac{c}{d} \cdot bd\right)}{bd}$ $(\because (a+b)c = ac + bc)$

$= \frac{\left(ad \cdot \frac{1}{b} \cdot b + cb \frac{1}{d} \cdot d\right)}{bd}$ $(\because ab = ba)$

$= \frac{(ad \cdot 1 + cb \cdot 1)}{bd}$ $(\because a \cdot \frac{1}{a} = 1)$

$= \frac{ad + bc}{bd}$ $(\because ab = b \cdot a)$

(Proved)

Q.5 Prove that $\frac{-7}{12} - \frac{5}{18} = \frac{-21-10}{36}$

$LHS = \left(\frac{-7}{12} - \frac{5}{18}\right)$

$= \left(\frac{-7}{12} - \frac{5}{18}\right) \cdot 1$ $(\because a \cdot 1 = a)$

$= \left(\frac{-7}{12} - \frac{5}{18}\right) 36 \cdot \frac{1}{36}$ $(\because a \cdot \frac{1}{a} = 1)$

$= \frac{\left(\frac{-7}{12} \cdot 36 - \frac{5}{18} \cdot 36\right)}{36}$ $(\because a(b-c) = ab - ac)$

$= \frac{\left(\frac{-7}{12} \cdot 12 \cdot 3 - \frac{5}{18} \cdot 18 \cdot 2\right)}{36}$

$= \frac{\left(-7 \cdot 3 \cdot 12 \cdot \frac{1}{12} - 5 \cdot 2 \cdot 18 \cdot \frac{1}{18}\right)}{36}$

$(\because a \cdot \frac{1}{a} = 1)$

$= \frac{(-7 \cdot 3 \cdot 1 - 5 \cdot 2 \cdot 1)}{36}$

$(\because a \cdot 1 = a)$

$= \frac{-21 - 10}{36}$ (Proved).

Q.6 Simplify by each Step Justification:

(i) $\frac{4 + 16x}{4}$

$= \frac{(4 \cdot 1 + 4 \cdot 4x)}{4}$

$= \frac{4 \cdot (1 + 4x)}{4}$ $\because a \cdot (b+c) = ab + ac$

$= 4 \cdot \frac{1}{4} (1 + 4x)$

$= 1 \cdot (1 + 4x)$ $(\because a \cdot 1/a = 1)$

$= 1 + 4x$ Ans. $(\because a \cdot 1 = a)$

(ii) $\frac{1/4 + 1/5}{1/4 - 1/5}$

Tahir Mahmood
M.Sc. (Math)
Mob No: 0345-6510779

$= \frac{(1/4 + 1/5) \cdot 1}{(1/4 - 1/5) \cdot 1}$ $(\because a \cdot 1 = a)$

$= \frac{\left(\frac{1}{4} + 1/5\right) \cdot 20 \cdot \frac{1}{20}}{\left(\frac{1}{4} - 1/5\right) 20 \cdot \frac{1}{20}}$ $(\because a \cdot 1/a = 1)$

$= \frac{\left(\frac{1}{4} \cdot 20 + \frac{1}{5} \cdot 20\right) \cdot 1/20}{\left(\frac{1}{4} \cdot 20 - \frac{1}{5} \cdot 20\right) 1/20}$ $(\because a(b \pm c) = ab \pm ac)$

$= \frac{\left(\frac{1}{4} \cdot 5 \cdot 4 + \frac{1}{5} \cdot 5 \cdot 4\right)}{\left(\frac{1}{4} \cdot 5 \cdot 4 - \frac{1}{5} \cdot 5 \cdot 4\right)} \quad 1/20 \cdot \frac{1}{1/20}$

$= \frac{\left(1 \cdot 5 \cdot 4 \cdot \frac{1}{4} + 4 \cdot \frac{1}{5} \cdot 5\right)}{\left(5 \cdot \frac{1}{4} \cdot 4 - 4 \cdot \frac{1}{5} \cdot 5\right)} \cdot 1$ $(\because a \cdot \frac{1}{a} = 1)$

Tahir Mahmood
M.Sc. (Math)
Mob No. 0345-6510779

$$= \frac{(5\cdot 1 + 4\cdot 1)}{(5\cdot 1 - 4\cdot 1)} \qquad (\because a\cdot \tfrac{1}{a} = 1)$$

$$= \frac{5+4}{5-4} = \frac{9}{1}$$

$$= 9 \quad \text{Ans.}$$

(iii) $\dfrac{\left(\frac{a}{b} + \frac{c}{d}\right)}{\left(\frac{a}{b} - \frac{c}{d}\right)}$

$$= \frac{\left(\frac{a}{b} + \frac{c}{d}\right)\cdot 1}{\left(\frac{a}{b} - \frac{c}{d}\right)\cdot 1} \qquad (\because a\cdot 1 = a)$$

$$= \frac{\left(\frac{a}{b} + \frac{c}{d}\right)\cdot bd \cdot \frac{1}{bd}}{\left(\frac{a}{b} - \frac{c}{d}\right)\cdot bd \cdot \frac{1}{bd}} \qquad (\because a\cdot \tfrac{1}{a} = 1)$$

$(\because a(b+c) = ab+ac)$

$$= \frac{\left(\frac{a}{b}\cdot bd + \frac{c}{d}\cdot bd\right)}{\left(\frac{a}{b}\cdot bd - \frac{c}{d}\cdot bd\right)} \quad \frac{1}{bd}\cdot\frac{1}{1/bd}$$

$$= \frac{\left(a\cdot d\cdot \frac{1}{b}\cdot b + c\cdot b\cdot \frac{1}{d}\cdot d\right)}{\left(a\cdot d\cdot b\cdot \frac{1}{b} - c\cdot b\cdot \frac{1}{d}\cdot d\right)}\cdot 1 \qquad (\because a\cdot \tfrac{1}{a} = 1) \quad (\because ab = ba)$$

$$= \frac{(ad\cdot 1 + cb\cdot 1)}{(ad\cdot 1 - cb\cdot 1)} \qquad (\because a\cdot \tfrac{1}{a} = 1)$$

$$= \frac{ad+cb}{ad-cb} \qquad (\because a\cdot 1 = a)$$

$$= \frac{ad+bc}{ad-bc} \qquad (\because a\cdot b = b\cdot a)$$

Ans.

(iv) $\dfrac{\frac{1}{a} - \frac{1}{b}}{1 - \frac{1}{a}\cdot\frac{1}{b}}$

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$$= \frac{\left(\frac{1}{a} - \frac{1}{b}\right)\cdot 1}{\left(1 - \frac{1}{a}\cdot\frac{1}{b}\right)\cdot 1} \qquad (\because a\cdot 1 = a)$$

$$= \frac{\left(\frac{1}{a} - \frac{1}{b}\right) ab\cdot \frac{1}{ab}}{\left(1 - \frac{1}{a}\cdot\frac{1}{b}\right) ab\cdot\frac{1}{ab}} \qquad (\because a\cdot \tfrac{1}{a} = 1)$$

$$= \frac{\left(\frac{1}{a}\cdot ab - \frac{1}{b}\cdot ab\right)}{\left(ab - \frac{1}{a}\cdot\frac{1}{b}\cdot ab\right)} \quad \frac{1}{ab}\cdot\frac{1}{1/ab}$$

$$= \frac{\left(\frac{1}{a}\cdot a\cdot b - \frac{1}{b}\cdot b\cdot a\right)}{\left(ab - \frac{1}{a}\cdot a\cdot \frac{1}{b}\cdot b\right)}\cdot 1 \qquad (\because a\cdot \tfrac{1}{a} = 1) \quad (\because ab = ba)$$

$\because a\cdot 1 = a$ and $a\cdot \frac{1}{a} = 1$

$$= \frac{1\cdot b - 1\cdot a}{(ab - 1\cdot 1)}$$

$(\because a\cdot 1 = a)$

$$= \frac{b-a}{ab-1} \quad \text{Ans.}$$

Complex Number:-

"The number of the form $a + ib$ is called complex number."

a is called real part and b is called imaginary part of complex number while i (iota) is called imaginary unit and $i = \sqrt{-1}$
$\Rightarrow i^2 = -1$, $i^3 = -i$, $i^4 = 1$

History of Complex numbers:

Year	Person	Event
1637	Descartes of France	Introduced terms real and imaginary.
1748	Euler of Switzerland	Used $i = \sqrt{-1}$
1832	Guass of Germany	Introduced the term complex number.