TAHIR MEHMOOD
M.Sc. Math
0345-6510779

Chapter: 1 (IInd Year)

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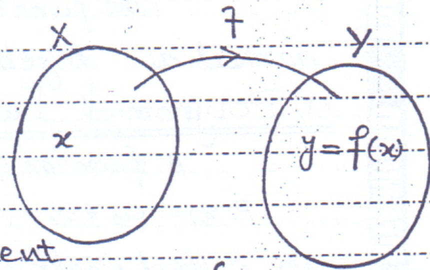
Fuctions and Limits

Function:-

Let X and Y be two non empty sets then a rule "f" is said to be function from X to Y if "f" assigns (links) every $x \in X$ uniquely with some element $y \in Y$. It is denoted by $f: X \to Y$ or $X \xrightarrow{f} Y$ or $y = f(x)$ (Euler's Notation of function) where x is independent variable and y is dependent variable. Function was introduced by Leibniz in 1694.



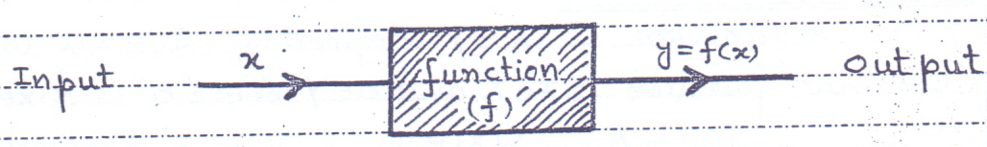
Domain and Range of Function:-

Let $f: X \to Y$ be a real valued function (from $R \to R$) then domain and range of "f" is denoted by D_f and R_f and are defined as:

$$D_f = \{x \in X \mid f(x) \text{ is defined}\}$$

$f(x)$ is defined if $f(x) \not\to \pm\infty$ or $f(x) \notin C$.

$$R_f = \{y \in Y \mid y = f(x) \text{ for } x \in X\}.$$



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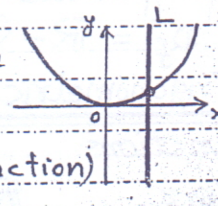
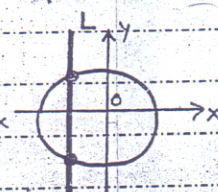
In fact, a function is a mechine in which independent variable is input and dependent variable is out put.

Vertical Line Test:-

If a vertical line cuts the graph at just one point then it is function otherwise not a function.



(Function)



Not function

Types of Functions:-

I. Algebraic Functions:-

The functions defined by Algebraic expressions are named as Algebraic functions.

(i) Polynomial Function:-

A function of the form

$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ where $a_0, a_1, a_2, \cdots, a_n$ are called coefficients and are real numbers. n is non-negative integer. If $a_n \neq 0$ then "n" is called degree of $P(x)$ and a_n is called Leading Coefficient.

e.g. $P(x) = 3x^4 + 7x - 19$ is polynomial of degree 4 and leading Coefficient 3.

(ii) Linear Function:-

A polynomial function of degree One is called Linear function. It is of the form $f(x) = ax + b$ where $a \neq 0$

Domain and range of Linear function is set of real numbers.

(iii) Identity Function:-

A function $I : X \to X$ defined by $I(x) = x \ \forall x \in X$ is called identity function. It may be denoted by $f(x) = x$

Identity function is bijective (one-one and onto) function.

(iv) Constant Function:-

A function $C : X \to Y$ defined by $C(x) = a$ is called Constant function $\forall x \in X$ and $a \in Y$ where "a" is fixed in Y.

e.g. $f(x) = \pi$, $C(x) = 10$, $g(x) = e$ are Constant functions.

(v) Rational Function:-

A function of the form $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ & $Q(x)$ are polynomials and $Q(x) \neq 0$ is called rational function.

$D_f = \mathbb{R} - \{x \mid Q(x) = 0\}$.

II. Trigonometric Functions:-

Functions	Domains	Ranges:
$y = \sin x$	$\mathbb{R} = (-\infty, \infty)$	$-1 \le y \le 1 = [-1, 1]$
$y = \cos x$	$\mathbb{R} = (-\infty, \infty)$	$-1 \le y \le 1 = [-1, 1]$
$y = \tan x$	$\mathbb{R} - \{x \mid x = (2n+1)\frac{\pi}{2}\}$, $n \in Z$	$\mathbb{R} = (-\infty, \infty)$

$y = \cot x$	$\mathbb{R} - \{x \mid x = n\pi,\ n \in Z\}$	$\mathbb{R} = (-\infty, \infty)$
$y = \sec x$	$\mathbb{R} - \{x \mid x = (2n+1)\frac{\pi}{2},\ n \in Z\}$	$y \leq -1 \wedge y \geq 1$
$y = \operatorname{cosec} x$	$\mathbb{R} - \{x \mid x = n\pi,\ n \in Z\}$	$y \leq -1 \wedge y \geq 1$

III. Inverse Trigonometric Functions:-

Functions	Domains	Ranges
$y = \sin^{-1} x \Leftrightarrow x = \sin y$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$y = \cos^{-1} x \Leftrightarrow x = \cos y$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1} x \Leftrightarrow x = \tan y$	$\mathbb{R} = (-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$y = \cot^{-1} x \Leftrightarrow x = \cot y$	$\mathbb{R} = (-\infty, \infty)$	$[-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$
$y = \sec^{-1} x \Leftrightarrow x = \sec y$	$x \leq -1 \wedge x \geq 1$	$[0, \pi] - \{\frac{\pi}{2}\}$
$y = \operatorname{cosec}^{-1} x \Leftrightarrow x = \operatorname{cosec} y$	$x \leq -1 \wedge x \geq 1$	$[-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$.

IV. EXPONENTIAL FUNCTIONS:-

A function inwhich variable appears as exponent (Power) is called exponential function. These are:

$y = a^x$ for $a > 0$ and $a \neq 1$ — Common Exponential Function
a = base of Function
Domain = $\mathbb{R}$
Range = Positive real Numbers (R^+)

$y = e^x$ — Natural Exponential function
e = Natural Base $\approx 2.7183\ldots$
Domain = $\mathbb{R}$
Range = $\mathbb{R}^+$

* Exponential function can never be zero or negative.

V. LOGARITHMIC FUNCTIONS:-

A function defined by
$y = \log_a x$ iff $x = a^y$
where $a > 0$ and $a \neq 1$
is called Common Logrithemic function

$y = \ln x$ iff $x = e^y$
is called natural exponential Function.

* If $a = 10$ then Function is called Briggs's logrithemic Function.
* Natural Logarithemic function is also called Naprian Logrithemic function

Domain = Positive real numbers = $\mathbb{R}^+$

Range = $\mathbb{R}$.

* Base of Logarithemic function should always be +ve & $\neq 1$.

VI Hyperbolic functions:-

Functions	Domains	Ranges.
$\sinh x = e^x - e^{-x}$	$\mathbb{R}$	$[0, \infty)$
$\cosh x = \frac{e^x + e^{-x}}{2}$	$\mathbb{R}$	$[1, \infty)$
$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$	$\mathbb{R}$	$[0, \infty)$
$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$	$\mathbb{R} - \{0\}$	$(0, \infty)$
$\operatorname{sech} x = \frac{2}{e^x + e^{-x}} = \frac{1}{\cosh x}$	$\mathbb{R}$	$(0, 1]$
$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$	$\mathbb{R} - \{0\}$	$(0, \infty)$

VII Inverse Hyperbolic functions:-

Functions	Domains	Ranges.
$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$	$\mathbb{R}$	$(0, \infty) = \mathbb{R}^+$
$\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$	$[1, \infty)$	$(0, \infty) = \mathbb{R}^+$
$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$	$(-1, 1)$	$(0, \infty) = \mathbb{R}^+$
$\coth^{-1} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right)$	$(-\infty, -1) \cup (1, \infty)$	$(0, \infty) = \mathbb{R}^+$
$\operatorname{sech}^{-1} x = \ln\left(\frac{1 + \sqrt{1-x^2}}{x}\right)$	$(0, 1]$	$(0, \infty) = \mathbb{R}^+$
$\operatorname{cosech}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{\lvert x \rvert}\right)$	$(-\infty, 0) \cup (0, \infty) = \mathbb{R} - \{0\}$.	$(0, \infty) = \mathbb{R}^+$

VIII EXPLICIT FUNCTION:-

A function of the form $y = f(x)$ is called explicit function.

e.g $y = \sin x$, $y = x^2 - 1$, $y = \log_a x$, $y = \tan^{-1} x$ etc.

IX Implicit function:-

A function of the form $f(x, y) = 0$ is called implicit function.

e.g. $\sin xy - x^2 + y^3 = 0$, $\ln(xy) + e^{-xy} + 1 = 0$

X Parametric Functions:-

A function of x and y is called parametric function if x and y are expressed as function of a third variable θ or t.

e.g $x = a\cos\theta$, $y = a\sin\theta$ for $x^2 + y^2 = a^2$.

XI Even Function:-

A function "f" is said to be even function if $f(-x) = f(x)$ where $x, -x \in D_f$.

e.g. $f(x) = x^2$ $\quad f(x) = \frac{1}{x^2}$

$f(x) = \cos x$ $\quad f(x) = \sec x$

$f(x) = \cosh x$ $\quad f(x) = \operatorname{sech} x$ are even functions.

XII Odd Function:-

A function "f" is said to be odd function if $f(-x) = -f(x)$ where $x, -x \in D_f$

e.g. $f(x) = x$ $\quad f(x) = x^3$

$f(x) = \sin x$ $\quad f(x) = \tan x$

$f(x) = \sinh x$ $\quad f(x) = \operatorname{cosech} x$ are odd functions.

Smooth and Piecewise (Sectional) function:-

A function $y = f(x)$ expressed by a single curve is called smooth function.

Such as $f(x) = \sin x$, $f(x) = x^2$

A function defined by sections (Peices of functions) is called Piecewise or Sectional function.

Such as:
$$f(x) = \begin{cases} x & \text{if } x \le 0 \\ x^2 - 1 & \text{if } 0 < x < 3 \\ \sin x & \text{if } 3 \le x \end{cases} \quad \text{etc.}$$

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Show that (i) $\cosh^2 x - \sinh^2 x = 1$ $\quad$ (ii) $\cosh^2 x + \sinh^2 x = \cosh 2x$.

(i)
$$LHS = \cosh^2 x - \sinh^2 x$$
$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$= \frac{e^{2x} + e^{-2x} + 2}{4} - \frac{e^{2x} + e^{-2x} - 2}{4}$$
$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{4}$$
$$= \frac{4}{4} = 1 = RHS.$$

Thus
$$\cosh^2 x - \sinh^2 x = 1 \quad \text{(Proved)}$$

(ii)
$$LHS = \cosh^2 x + \sinh^2 x$$
$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$= \frac{e^{2x} + e^{-2x} + 2}{4} + \frac{e^{2x} + e^{-2x} - 2}{4}$$
$$= \frac{e^{2x} + e^{-2x} + 2 + e^{2x} + e^{-2x} - 2}{4}$$
$$= \frac{2[e^{2x} + e^{-2x}]}{4} = \frac{e^{2x} + e^{-2x}}{2}$$
$$= \cosh 2x = RHS.$$

Thus $\cosh^2 x + \sinh^2 x = \cosh 2x$ (Proved)

Exercise: 1.1

Q.1 Given that:

(a) $f(x) = x^2 - x$

(i) $f(-2) = ?$

$f(-2) = (-2)^2 - (-2)$

$f(-2) = 4 + 2 = 6$ Ans.

(ii) $f(0) = ?$

$f(0) = (0)^2 - (0) = 0 - 0$

$f(0) = 0$ Ans.

(iii) $f(x-1) = ?$

$f(x-1) = (x-1)^2 - (x-1)$

$f(x-1) = x^2 + 1 - 2x - x + 1$

$f(x-1) = x^2 - 3x + 2$ Ans.

(iv) $f(x^2+4) = ?$

$f(x^2+4) = (x^2+4)^2 - (x^2+4)$

$f(x^2+4) = x^4 + 16 + 8x^2 - x^2 - 4$

$f(x^2+4) = x^4 + 7x^2 + 12$ Ans.

(b) $f(x) = \sqrt{x+4}$

(i) $f(-2) = ?$

$f(-2) = \sqrt{(-2)+4} = \sqrt{-2+4}$

$f(-2) = \sqrt{2}$ Ans.

(ii) $f(0) = ?$

$f(0) = \sqrt{0+4} = \sqrt{4}$

$f(0) = 2$ Ans.

(iii) $f(x-1) = ?$

$f(x-1) = \sqrt{(x-1)+4}$

$f(x-1) = \sqrt{x+3}$ Ans.

(iv) $f(x^2+4) = ?$

$f(x^2+4) = \sqrt{(x^2+4)+4}$

$f(x^2+4) = \sqrt{x^2+8}$ Ans.

Q.2 Find $\dfrac{f(a+h) - f(a)}{h} = ?$

(i) $f(x) = 6x - 9$

$f(a+h) = 6(a+h) - 9$

$f(a+h) = 6a + 6h - 9$

$f(a) = 6a - 9$

$\dfrac{f(a+h) - f(a)}{h} = \dfrac{(6a+6h-9) - (6a-9)}{h}$

$= \dfrac{6a+6h-9-6a+9}{h}$

$= \dfrac{6h}{h} = 6$

$\dfrac{f(a+h) - f(a)}{h} = 6$ Ans.

(ii) $f(x) = \sin x$

$f(a+h) = \sin(a+h)$

$f(a) = \sin a$

$\dfrac{f(a+h) - f(a)}{h} = \dfrac{\sin(a+h) - \sin a}{h}$

$= \dfrac{2\cos\left(\frac{a+h+a}{2}\right)\sin\left(\frac{a+h-a}{2}\right)}{h}$

$= \dfrac{2}{h}\cos\left(\dfrac{2a+h}{2}\right)\sin\dfrac{h}{2}$ Ans.

$= \dfrac{2}{h}\cos\left(a + \dfrac{h}{2}\right)\sin\dfrac{h}{2}$ Ans.

$\because \sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

(iii) $f(x) = x^3 + 2x^2 - 1$

$f(a) = a^3 + 2a^2 - 1$

$f(a+h) = (a+h)^3 + 2(a+h)^2 - 1$

$= (a^3+h^3+3a^2h+3ah^2) + 2(a^2+h^2+2ah) - 1$

$= a^3+h^3+3a^2h+3ah^2+2a^2+2h^2+4ah-1$

$\frac{f(a+h)-f(a)}{h} = \frac{(a^3+h^3+3a^2h+3ah^2+2a^2+2h^2+4ah-1)-(a^3+2a^2-1)}{h}$

$= \frac{a^3+h^3+3a^2h+3ah^2+2a^2+2h^2+4ah-1-a^3-2a^2+1}{h}$

$= \frac{h^3+3a^2h+3ah^2+2h^2+4ah}{h}$

$= \frac{h(h^2+3a^2+3ah+2h+4a)}{h}$

$= h^2+3a^2+3ah+2h+4a$

$= h^2+(3a+2)h+(3a^2+4a)$ Ans.

(iv) $f(x) = \cos x$

$f(a) = \cos a$

$f(a+h) = \cos(a+h)$

$\frac{f(a+h)-f(a)}{h} = \frac{\cos(a+h) - \cos a}{h}$

$= \frac{-2\sin\left(\frac{a+h+a}{2}\right)\sin\left(\frac{a+h-a}{2}\right)}{h}$

$\because \cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$= \frac{-2}{h}\sin\left(\frac{2a+h}{2}\right)\sin\frac{h}{2}$

$= \frac{-2}{h}\sin\left(a+\frac{h}{2}\right)\sin\frac{h}{2}$ Ans.

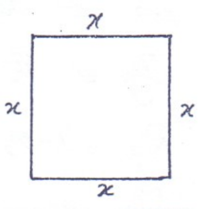
Q.3 Express the following:

(i) Perimeter P of Square as function of area.

Let x be the side of square

Area of Square (A) $= x^2$

$A = x^2$ —— (i)



Perimeter of Square (P)

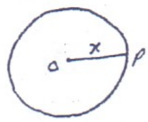
$P = 4x$ —— (ii)

From (i) $x = \sqrt{A}$

(2) $\Rightarrow$ $P(A) = 4\sqrt{A}$ Ans.

(ii) Area of Circle as function of Circumference

Let x be the radius of circle then



Circumference (C) $= 2\pi x$ —— (i)

Area of Circle (A) $= \pi x^2$ —— (ii)

From (i) $x = \frac{C}{2\pi}$

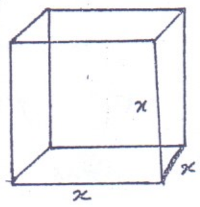
(ii) $\Rightarrow$ $A(C) = \pi\left(\frac{C}{2\pi}\right)^2$

$A(C) = \pi \cdot \frac{C^2}{4\pi^2}$

$A(C) = \frac{1}{4\pi} C^2$ Ans.

(iii) Volume V of Cube as function of base area (A).

Let x be the each side of Cube.



Area of base (A) $= x \cdot x$

$A = x^2$ —— (i)

Volume of Cube (V) $= x \cdot x \cdot x$

$V = x^3$ —— (ii)

From (i) $\sqrt{A} = x \Rightarrow x = A^{1/2}$

(ii) $\Rightarrow$ $V(A) = (A^{1/2})^3$

$V(A) = A^{3/2}$ Ans.

* Volume of Cube = Length × Width × height
* Area of Square = Side × Side
* Perimeter of Square = 4 × Side
* Area of Circle = $\pi \times (\text{Radius})^2$
* Circumference of circle = $2\pi \times (\text{Radius})$.

Q.4 Find domain and range and also sketch the graph of g as followed:

(i) $g(x) = 2x - 5$

For every value of real numbers $g(x)$ is real valued function so

D_g = Set of real numbers ($\mathbb{R}$)

R_g = Set of real numbers ($\mathbb{R}$)

Graph of $g(x) = 2x - 5$

x	-4	-3	-2	-1	0	1	2	3	4
$g(x) = y$	-13	-11	-9	-7	-5	-3	-1	1	3

(ii) $g(x) = \sqrt{x^2 - 4}$

For $x^2 < 4$ $g(x)$ is imaginary

so $x \geq +2$ and $x \leq -2$, Thus

$D_g = \mathbb{R} - (-2, 2)$

$R_g = [0, \infty)$

Graph of $g(x) = \sqrt{x^2 - 4}$

x	-5	-4	-3	-2	2	3	4	5
$g(x)$	$\sqrt{21}$	$\sqrt{12}$	$\sqrt{5}$	0	0	$\sqrt{5}$	$\sqrt{12}$	$\sqrt{21}$

(iii) $g(x) = \sqrt{x+1}$

$g(x)$ becomes imaginary for $x < -1$, Thus

$D_g = [-1, \infty[$ or $\mathbb{R} - (-\infty, -1)$

$R_g = [0, +\infty)$ or $\mathbb{R} - (-\infty, 0)$

The graph of $g(x) = \sqrt{x+1}$

x	0	1	2	3	4	5	6	7
$g(x)$	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{5}$	$\sqrt{6}$	$\sqrt{7}$	$2\sqrt{2}$

(iv) $g(x) = |x - 3|$

Absolute valued function is always real valued function for every $x \in X$

Thus $D_g = \mathbb{R}$ or $]-\infty, +\infty[$

$R_g = [0, +\infty[$

The graph of $g(x) = |x - 3|$

x	-3	-2	-1	0	1	2	3	4
$g(x)$	6	5	4	3	2	1	0	1

(v) $g(x) = \begin{cases} 6x+7 & \text{if } x \leq -2 \\ 4x-3 & \text{if } x > -2 \end{cases}$

$g(x) = 6x + 7$ for $x \leq -2$ and is real valued for all the values of x also $g(x) = 4x - 3$ for $x > -2$ is real valued for all the values of x, Thus

$D_g = \mathbb{R}$ or $]-\infty, +\infty[$

$R_g =]-\infty, -5] \cup]-7, \infty[\Rightarrow]-\infty, \infty[$

The graph of $g(x) = \begin{cases} 6x+7 & x \leq -2 \\ 4x-3 & x > -2 \end{cases}$

x	-4	-3	-2	-1	0	1	2	3
$g(x)$	-17	-11	-5	-7	-3	1	5	9

Tahir Mahmood
M.Sc. (Math)
Mob No: 0[illegible]10779

(vi) $g(x) = \begin{cases} x-1 & , x < 3 \\ 2x+1 & , x \geq 3 \end{cases}$

$g(x) = x-1$ for $x < 3$ is real valued for all the values of x, also $g(x) = 2x+1$ for $x \geq 3$ is real valued for all value of x so

$D_g = \mathbb{R}$ or $]-\infty, +\infty[$

$R_g =]-\infty, 2[\cup [7, \infty[$

or $R_g = \mathbb{R} -]2, 7[$

The graph of $g(x) = \begin{cases} x-1 & , x<3 \\ 2x+1 & , x \geq 3 \end{cases}$

	$\overbrace{\quad}^{g(x) = x-1}$				$\overbrace{\quad}^{g(x) = 2x+1}$			
x	-1	0	1	2	3	4	5	6
$g(x)$	-2	-1	0	1	7	9	11	13

(vii) $g(x) = \dfrac{x^2+3x+2}{x+1}$

Clearly $g(x)$ becomes ∞ if $x = -1$ thus $g(x)$ will be real valued for all values of x other than $x = -1$

Thus $D_g = \mathbb{R} - \{-1\}$

$R_g = \mathbb{R} - \{1\}$

$\because$ when $x = -1$ $g(x) = 1$ (not possible)

The graph of $g(x) = \dfrac{x^2+3x+2}{x+1}$

x	-5	-4	-3	-2	0	1	2	3
$g(x)$	-3	-2	-1	0	2	3	4	5

(viii) $g(x) = \dfrac{x^2-16}{x-4}$

Clearly $g(x)$ becomes ∞ if $x = 4$

$g(x)$ will be real valued if $x \neq 4$, so

$D_g = \mathbb{R} - \{4\}$

$R_g = \mathbb{R} - \{8\}$

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$\because$ when $x = 4$ $g(x) = 8$ (Not possible)

Thus graph of $g(x) = \dfrac{x^2-16}{x-4}$

x	0	1	2	3	5	6	7	8
$g(x)$	4	5	6	7	9	10	11	12

Q.5 $f(x) = x^3 - ax^2 + bx + 1$

$f(2) = -3$, $f(-1) = 0$ $\quad a, b = ?$

$f(2) = (2)^3 - a(2)^2 + b(2) + 1$

$f(2) = 8 - 4a + 2b + 1$

$-3 = 9 - 4a + 2b \quad \because f(2) = -3$

$4a - 2b = 9 + 3$

$4a - 2b = 12 \Rightarrow 2a - b = 6$ ___ (i)

$f(-1) = (-1)^3 - a(-1)^2 + b(-1) + 1$

$f(-1) = -1 - a - b + 1$

$0 = -a - b \Rightarrow -(a+b) = 0 \quad \because f(-1) = 0$

$a + b = 0$ ___ (ii)

Adding (i) and (ii)

$3a = 6 \Rightarrow \boxed{a = 2}$

Putting in (ii) $2 + b = 0 \Rightarrow \boxed{b = -2}$

Q.6 $h = 60\ m$

$$h(x) = 40 - 10x^2$$

(i) What is the height of stone when

(a) $x = 1$ sec

$$h(1) = 40 - 10(1)^2$$

$$h(1) = 40 - 10 = 30m \quad \text{Ans.}$$

(b) $x = 1.5$ sec

$$h(1.5) = 40 - 10(1.5)^2$$

$$h(1.5) = 40 - 10(2.25) \Rightarrow 40 - 22.5$$

$$h(1.5) = 17.5\ m \quad \text{Ans.}$$

(c) $x = 1.7$ sec

$$h(1.7) = 40 - 10(1.7)^2$$

$$h(1.7) = 40 - 10(2.89) \Rightarrow 40 - 28.9$$

$$h(1.7) = 11.1\ m \quad \text{Ans.}$$

(ii) $x = ?$ $\quad h = 0\ m$ (at ground)

$$0 = 40 - 10x^2$$

$$0 - 40 = -10x^2 \Rightarrow 10x^2 = 40$$

$$\frac{40}{10} = x^2 \Rightarrow x^2 = 4$$

$$x = 2 \text{ sec} \quad \text{Ans.}$$

Q.7 Show that:

(i) $x = at^2$ —ⓐ $\quad y = 2at$ —ⓑ

ⓑ $\Rightarrow y = 2at \Rightarrow t = y/2a$

Putting in ⓐ $\quad x = a\left[y/2a\right]^2$

$$x = a\,\frac{y^2}{4a^2}$$

$$\Rightarrow 4ax = y^2 \quad \text{(Proved).}$$

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(ii) $x = a\cos\theta \quad y = b\sin\theta$

$$\frac{x}{a} = \cos\theta \qquad \frac{y}{b} = \sin\theta$$

Squaring and adding

$$\left(\frac{x}{a}\right)^2 + (y/b)^2 = \cos^2\theta + \sin^2\theta$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\because \cos^2\theta + \sin^2\theta = 1) \text{ (Proved).}$$

(iii) $x = a\sec\theta \qquad y = b\tan\theta$

$$\frac{x}{a} = \sec\theta \qquad \frac{y}{b} = \tan\theta$$

Squaring and Subtracting

$$\left(\frac{x}{a}\right)^2 - (y/b)^2 = \sec^2\theta - \tan^2\theta$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ (Proved).}$$

$$\because \left\{ \begin{array}{l} 1 + \tan^2\theta = \sec^2\theta \\ 1 = \sec^2\theta - \tan^2\theta \end{array} \right.$$

Q.8 Prove the identities:

(i) $\sinh 2x = 2\sinh x\cosh x$

RHS $= 2\sinh x \cosh x$

$$= \cancel{2}\left(\frac{e^x - e^{-x}}{\cancel{2}}\right)\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{e^{2x} - e^{-2x}}{2}$$

$$= \sinh 2x \quad \text{(Proved)}$$

$$\because \left\{ \begin{array}{l} \sinh x = \frac{e^x - e^{-x}}{2} \\ \cosh x = \frac{e^x + e^{-x}}{2} \end{array} \right.$$

(ii) $\operatorname{Sech}^2 x = 1 - \tanh^2 x$

RHS $= 1 - \tanh^2 x$

$$= 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2$$

$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$= \frac{(e^{2x} + e^{-2x} + 2) - (e^{2x} + e^{-2x} - 2)}{(e^x + e^{-x})^2}$$

$$\because \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

$$= \frac{e^{2x}+e^{-2x}+2-e^{2x}+2-e^{-2x}}{(e^x+e^{-x})^2}$$

$$= \frac{4}{(e^x+e^{-x})^2} \Rightarrow \frac{(2)^2}{(e^x+e^{-x})^2}$$

$$= \left[\frac{2}{e^x+e^{-x}}\right]^2 = \operatorname{sech}^2 x \text{ (Proved)}$$

(iii) $\operatorname{Cosech}^2 x = \operatorname{Coth}^2 x - 1$

$\because \operatorname{Cosech} x = \frac{2}{e^x-e^{-x}}$, $\operatorname{Coth} x = \frac{e^x+e^{-x}}{e^x-e^{-x}}$

RHS $= \operatorname{Coth}^2 x - 1$

$$= \left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)^2 - 1$$

$$= \frac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x-e^{-x})^2}$$

$$= \frac{(e^{2x}+e^{-2x}+2)-(e^{2x}+e^{-2x}-2)}{(e^x-e^{-x})^2}$$

$$= \frac{e^{2x}+e^{-2x}+2-e^{2x}-e^{-2x}+2}{(e^x-e^{-x})^2}$$

$$= \frac{4}{(e^x-e^{-x})^2} \Rightarrow \frac{(2)^2}{(e^x-e^{-x})^2}$$

$$= \left(\frac{2}{e^x-e^{-x}}\right)^2 = \operatorname{Cosech}^2 x \text{ (Proved).}$$

Q.9 Determine whether f is even or odd.

(i) $f(x) = x^3+x$

> 1) The function will be even if
> $f(x) = f(-x)$
> 2) The function will be odd if
> $f(-x) = -f(x)$

$f(-x) = (-x)^3+(-x)$

$f(-x) = -x^3-x$

$f(-x) = -(x^3+x)$

$f(-x) = -f(x)$

Thus $f(x) = x^3+x$ is odd function.

(ii) $f(x) = (x+2)^2$

$f(-x) = \{(-x)+2\}^2$

$f(-x) = (-x+2)^2$

$f(-x) \neq f(x) \neq -f(x)$

Thus $f(x)$ is neither odd nor Even.

(iii) $f(x) = x\sqrt{x^2+5}$

$f(-x) = (-x)\sqrt{(-x)^2+5}$

$f(-x) = -x\sqrt{x^2+5}$

$f(-x) = -f(x)$

Thus $f(x)$ is Odd function.

(iv) $f(x) = \frac{x-1}{x+1}$

$f(-x) = \frac{(-x)-1}{(-x)+1} = \frac{-x-1}{-x+1}$

$f(-x) \neq -f(x) \neq f(x)$

Thus $f(x)$ is neither even nor odd.

(v) $f(x) = x^{2/3}+6$

$f(-x) = (-x)^{2/3}+6$

$f(-x) = x^{2/3}+6$

$f(-x) = f(x)$

Thus $f(x)$ is Even function.