CHAPTER

14 Solutions of Trignometric Equation

14.1 Introduction

The Equations, containing at least one trigonometric function, are called **Trigonometric Equations**, *e.g.*, each of the following is a trigonometric equation:

$$\sin x = \frac{2}{5},\ \ \text{Sec}\,x = \tan x \quad \text{and} \quad \sin^2 x - \sec x + 1 = \frac{3}{4}$$

Trigonometric equations have an infinite number of solutions due to the periodicity of the trigonometric functions. For example

$$\text{If } \sin\theta = \theta \text{ then } \theta = 0, \pm\ \ , \pm 2\ \ , \ldots$$

which can be written as $\theta = n\ \ $, where $n \in Z$.

In solving trigonometric equations, first find the solution over the interval whose length is equal to its period and then find the general solution as explained in the following examples:

Example 1: Solve the equation $\sin x = \frac{1}{2}$

Solution: $\sin x = \frac{1}{2}$

$\because$ sin x is positive in I and II Quadrants with the reference angle $x = \frac{\ }{6}$.

$\therefore\ x = \frac{\ }{6}$ and $x = \ \ - \frac{\ }{6} = \frac{5\ }{6}$, where $x \in [0, 2\ \ ]$

$\therefore$ General values of x are $\frac{\ }{6} + 2n\ \ $ and $\frac{5\ }{6} + 2n\ \ $, $n \in Z$

Hence solution set $= \left\{\frac{\ }{6} + 2n\ \ \right\} \cup \left\{\frac{5\ }{6} + 2n\ \ \right\}$, $n \in Z$

Example 2: Solve the equation: $1 + \cos x = 0$

Solution: $1 + \cos x = 0$

$\Rightarrow \quad \cos x = -1$

Since $\cos x$ is –ve, there is only one solution $x = \pi$ in $[0, 2\pi]$

Since 2π is the period of $\cos x$

$\therefore$ General value of x is $\pi + 2n\pi$, $\quad n \in Z$

Hence solution set = $\{\pi + 2n\pi\}$, $\quad n \in Z$

Example 3: Solve the equation: $4\cos^2 x - 3 = 0$

Solution: $4\cos^2 x - 3 = 0$

$\Rightarrow \quad \cos^2 x = \frac{3}{4} \quad \Rightarrow \quad \cos x = \pm\frac{\sqrt{3}}{2}$

i. If $\cos x = \frac{\sqrt{3}}{2}$

Since $\cos x$ is +ve in I and IV Quadrants with the reference angle

$x = \frac{\ }{6}$

$\therefore \; x = \frac{\ }{6}$ and $x = 2\ \ - \frac{\ }{6} = \frac{11}{6}$ $\qquad$ where $x \in [0, 2\ \ ]$

As 2π is the period of $\cos x$.

$\therefore$ General value of x are $\frac{\ }{6} + 2n$ and $\frac{11}{6} + 2n$, $\quad n \in Z$

ii. if $\cos x = -\frac{\sqrt{3}}{2}$

Since $\cos x$ is –ve in II and III Quadrants with reference angle $x = \frac{\ }{6}$

$\therefore \; x = \ \ - \frac{\ }{6} = \frac{5}{6}$ and $x = x + \frac{\ }{6} = \frac{7}{6}$ $\qquad$ where $x \in [0, 2\ \ ]$

As 2π is the period of $\cos x$.

$\therefore$ General values of x are $\frac{5}{6}+2n$ and $\frac{7}{6}+2n$, $n\in Z$

Hence solution set $=\left\{\frac{}{6}+2n\right\}\cup\left\{\frac{11}{6}+2n\right\}\cup\left\{\frac{5}{6}+2n\right\}\cup\left\{\frac{7}{6}+2n\right\}$

14.2 Solution of General Trigonometric Equations

When a trigonometric equation contains more than one trigonometric functions, trigonometric identities and algebraic formulae are used to transform such trigonometric equation to an equivalent equation that contains only one trigonometric function.

The method is illustrated in the following solved examples:

Example 1: Solve: sin x + cos x = 0.

Solution: sin x + cos x = 0

$$\Rightarrow \frac{\sin x}{\cos x}+\frac{\cos x}{\cos x}=0 \qquad (\text{Dividing by } \cos x \neq 0)$$

$$\Rightarrow \tan x+1=0 \qquad \Rightarrow \tan x=-1$$

$\because$ tan x is –ve in II and IV Quadrants with the reference angle

$x=\frac{}{4}$

$\therefore$ $x=-\frac{}{4}=\frac{3}{4}$, where $x\in[0,]$

As π is the period of tan x,

$\therefore$ General value of x is $\frac{3}{4}+n$, $n\in Z$

$\therefore$ Solution set = $\left\{\frac{3}{4}+n\right\}$, $n\in Z$.

Example 2: Find the solution set of: $\sin x\cos x=\frac{\sqrt{3}}{4}.$

Solution: $\sin x\cos x=\frac{\sqrt{3}}{4}.$

$\Rightarrow \quad \frac{1}{2}(2\sin x\cos x)=\frac{\sqrt{3}}{4}$

$\Rightarrow \quad \sin 2x=\frac{\sqrt{3}}{2}$

$\because$ sin $2x$ is +ve in I and II Quadrants with the reference angle $2x=\frac{\ }{3}$

$\therefore$ $2x=\frac{\ }{3}$ and $2x=\ -\frac{\ }{3}=\frac{2}{3}$ are two solutions in $[0,2\ \ ]$

As 2π is the period of sin $2x$.

$\therefore$ General values of $2x$ are $\frac{\ }{3}+2n$ and $\frac{2}{3}+2n$, , $n\in Z$

$\Rightarrow$ General values of x are $\frac{\ }{6}+n$ and $\frac{\ }{3}+n$, $n\in Z$

Hence solution set = $=\left\{\frac{\ }{6}+n\ \right\}\cup\left\{\frac{\ }{3}+n\ \right\}$, $n\in Z$

Note: In solving the equations of the form sin *kx* = *c*, we first find the solution pf sin *u* = *c* (where *kx* = *w*) and then required solution is obtained by dividing each term of this solution set by *k*.

Example 3: Solve the equation: sin $2x$ = cos $2x$

Solution: sin$2x$ = cos$2x$

$\Rightarrow$ 2sin$x$ cos x = cos$x$

$\Rightarrow$ 2sin$x$ cos x – cos$x$ = 0

$\Rightarrow$ cos$x$(2sin$x$ – 1) = 0

$\therefore$ $\cos x = 0$ or $2\sin x - 1 = 0$

i. If $\cos x = 0$

$\Rightarrow$ $x = \frac{}{2}$ and $x = \frac{3}{2}$ where $x \in [0, 2\pi]$

As 2π is the period of $\cos x$.

$\therefore$ General values of x are $\frac{\pi}{2} + 2n\pi$ and $\frac{3\pi}{2} + 2n\pi$, $n \in Z$,

ii. If $2 \sin x - 1 = 0$

$\Rightarrow$ $\sin x = \frac{1}{2}$

Since $\sin x$ is +ve in I and II Quadrants with the reference angle $x = \frac{\pi}{6}$

$\therefore$ $x = \frac{\pi}{6}$ and $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ where $x \in [0, 2\pi]$

As 2π is the period of $\sin x$.

$\therefore$ General values of x are and $\frac{\pi}{6} + 2n\pi$ and $5\frac{\pi}{6} + 2n\pi$, $n \in Z$,

Hence solution set = $\left[\frac{\pi}{2} + 2n\pi\right] \cup \left\{\frac{3\pi}{2} + 2n\pi\right\} \cup \left\{\frac{\pi}{6} + 2n\pi\right\} \cup \left\{5\frac{\pi}{6} + 2n\pi\right\}$,

$n \in z$

Example 4: Solve the equation: $\sin^2 x + \cos x = 1$.

Solution: $\sin^2 x + \cos x = 1$

$\Rightarrow$ $1 - \cos^2 x + \cos x = 1$

$\Rightarrow$ $-\cos x (\cos x - 1) = 0$

$\Rightarrow$ $\cos x = 0$ or $\cos x - 1 = 0$

i. If $\cos x = 0$

$\Rightarrow x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, where $x \in [0, 2\pi]$

As 2π is the period of $\cos x$

$\therefore$ General values of x are $\frac{\pi}{2} + 2n\pi$ and $\frac{3\pi}{2} + 2n\pi$, $n \in Z$

ii. If $\cos x = 1$

$\Rightarrow x = 0$ and $x = 2\pi$, where $x \in [0, 2\pi]$

As 2π is the period of $\cos x$

$\therefore$ General values of x are $0 + 2n\pi$ and $2\pi + 2n\pi$, $n \in Z$.

$$\therefore \text{Solution Set} = \left\{\frac{\pi}{2} + 2n\pi\right\} \cup \left\{\frac{3\pi}{2} + 2n\pi\right\} \cup \{2n\pi\} \cup \{2\pi + 2n\pi\}, n \in z$$

$$\because \{2(n+1)\pi\} \subset \{2n\pi\},\ n \in z$$

$$\text{Hence the solution set} = \left[\frac{\pi}{2} + 2n\pi\right] \cup \left\{\frac{3\pi}{2} + 2n\pi\right\} \cup \{2n\pi\}, n \in z$$

Sometimes it is necessary to square both sides of a trigonometric equation. In such a case, extaneous roots can occur which are to be discarded. So each value of x must be checked by substituting it in the given equation.

For example, $x = 2$ is an equation having a root 2. On squaring we get $x^2 - 4$ which gives two roots 2 and -2. But the root -2 does not satisfy the equation $x = 2$. Therefore, -2 is an extaneous root.

Example 5: Solve the equation: $\csc x = \sqrt{3} + \cot x$.

Solution: $\csc x = \sqrt{3} + \cot x$ ....(i)

$$\Rightarrow \frac{1}{\sin x} = \sqrt{3} + \frac{\cos x}{\sin x}$$

$$\Rightarrow 1 = \sqrt{3}\sin x + \cos x$$

$$\Rightarrow 1 - \cos x = \sqrt{3}\sin x$$

$$\Rightarrow (1 - \cos x)^2 = (\sqrt{3}\sin x)^2$$

$\Rightarrow 1-2\cos x+\cos^2 x=3\sin^2 x$

$\Rightarrow 1-2\cos x+\cos^2 x=3(1-\cos^2 x)$

$\Rightarrow 4\cos^2 x-2\cos x-2=0$

$\Rightarrow 2\cos^2 x-\cos x-1=0$

$\Rightarrow (2\cos x+1)(\cos x-1)=0$

$\Rightarrow \cos x=-\frac{1}{2}$ or $\cos x=1$

i. If $\cos x=-\frac{1}{2}$

Since cos x is –ve in II and III Quadrants with the reference angle $x=\frac{\pi}{3}$

$\Rightarrow x=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$ and $x=\pi+\frac{\pi}{3}=\frac{4\pi}{3}$, where $x\in [0, 2\pi]$

Now $x=\frac{4\pi}{3}$ does not satisfy the given equation (i).

$\therefore x=\frac{4\pi}{3}$ is not admissible and so $x=\frac{2\pi}{3}$ is the only solution.

Since 2π is the period of cos x

$\therefore$ General value of x is $\frac{2\pi}{3}+2n\pi$, $n\in Z$

ii. If cos x = 1

$\Rightarrow x=0$ and $x=2\pi$ where $x\in [0, 2\pi]$

Now both csc x and cot x are not defined for $x=0$ and $x=2$

$\therefore x=0$ and $x=2$ are not admissible.

Hence solution set = $\left\{\frac{2\pi}{3}+2n\pi\right\}$, $n\in Z$

Exercise 14

1. Find the solutions of the following equations which lie in $[0, 2\pi]$

i) $\sin x = -\frac{\sqrt{3}}{2}$ ii) $\text{cosec}\theta = 2$ iii) $\sec x = -2$ iv) $\cot\theta = \frac{1}{\sqrt{3}}$

2. Solve the following trigonometric equations:

i) $\tan^2\theta = \frac{1}{3}$ ii) $\text{cosec}^2\theta = \frac{4}{3}$ iii) $\sec^2\theta = \frac{4}{3}$ iv) $\cot^2\theta = \frac{1}{3}$

Find the values of θ satisfying the following equations:

3. $3\tan^2\theta + 2\sqrt{3}\tan\theta + 1 = 0$

4. $\tan^2\theta - \sec\theta - 1 = 0$

5. $2\sin\theta + \cos^2\theta - 1 = 0$

6. $2\sin^2\theta - \sin\theta = 0$

7. $3\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta - 3\sin^2\theta = 0$ [Hint: Divide by $\sin^2\theta$]

Find the solution sets of the following equations:

8. $4\sin^2\theta - 8\cos\theta + 1 = 0$

9. $\sqrt{3}\tan x - \sec x - 1 = 0$

10. $\cos 2x = \sin 3x$ [Hint: $\sin 3x = 3\sin x - 4\sin^3 x$]

11. $\sec 3\theta = \sec\theta$

12. $\tan 2\theta + \cot\theta = 0$

13. $\sin 2x + \sin x = 0$

14. $\sin 4x - \sin 2x = \cos 3x$

15. $\sin x + \cos 3x = \cos 5x$

16. $\sin 3x + \sin 2x + \sin x = 0$

17. $\sin 7x - \sin x = \sin 3x$

18. $\sin x + \sin 3x + \sin 5x = 0$

19. $\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta = 0$

20. $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$