Circles related to triangle:-

There are Three types of a Circle subject to triangle:

(i) Circumcircle.

(ii) Inscribed Circle.

(iii) Escribed Circle.

CIRCUMCIRCLE.

The circle which passes through vertices of a triangle is called Circumcircle. Its centre is called Circumcentre and radius is called Circum radius denoted by R.

* Circumcentre is the point of intersection of right bisectors of a triangle.

Show that

$$R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}.$$

Proof:- Consider three different triangle with $m\angle A = \alpha$

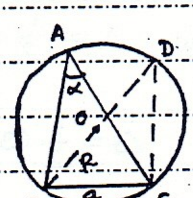
In Case of Acute triangle:-

Clearly $BD = 2R$

In ΔBCD

$m\angle BCD = 90°$

and $\sin\angle BDC = \frac{\overline{BC}}{\overline{BD}}$



$\Rightarrow \sin\alpha = \frac{a}{2R}$

$\because$ Angles of same segment are equal

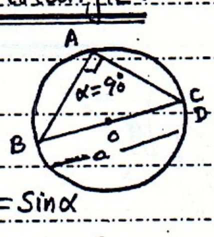
so $m\angle\alpha = m\angle BDC$

$\Rightarrow R = \frac{a}{2\sin\alpha}$

In Case of Right triangle:-

$\because BD = 2R$

$\Rightarrow \frac{\overline{BC}}{\overline{BD}} = \frac{\overline{BC}}{\overline{BC}} = 1$



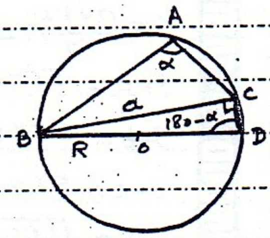
$\Rightarrow \frac{\overline{BC}}{\overline{BD}} = \sin 90° = \sin\alpha$

$\sin\alpha = \frac{a}{2R} \Rightarrow R = \frac{a}{2\sin\alpha}$

In Case of Abtuse Angled triangle:-

$m\angle BDC = 180 - \alpha$

Now ΔBCD



$\sin(180-\alpha) = \frac{\overline{BC}}{\overline{BD}}$

$\sin\alpha = \frac{a}{2R}$

$\Rightarrow R = \frac{a}{2\sin\alpha}$

Thus in each case $R = \frac{a}{2\sin\alpha}$

but $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$

so $R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}$

OR

$$2R = \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Show that $R = \frac{abc}{4\Delta}$

Proof:-

Consider

$R = \frac{a}{2\sin\alpha}$

$\because \sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

$\Rightarrow R = \frac{a}{2\left[2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right]}$

$R = \frac{a}{4\sqrt{\frac{(S-b)(S-c)}{bc}}\sqrt{\frac{S(S-a)}{bc}}}$

$\because \sin\frac{\alpha}{2} = \sqrt{\frac{(S-b)(S-c)}{bc}}$

$\cos\frac{\alpha}{2} = \sqrt{\frac{S(S-a)}{bc}}$

so $R = \frac{abc}{4\sqrt{S(S-a)(S-b)(S-c)}}$

$\because \Delta = \sqrt{S(S-a)(S-b)(S-c)}$

so $R = \frac{abc}{4\Delta}$ (Proved)

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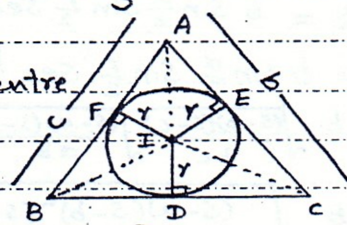
Inscribed Circle.

The Circle which touches the three sides of a triangle internally is called In-Circle. Its centre is called In-Centre and radius is called in-radius denoted by r.

* In-Centre is the point of intersection of the angle bisectors of a triangle.

Show that $r = \dfrac{\Delta}{S}$

Proof:-



Let I be in-centre. Draw ⊥lars from in-centre to each side.

Now $\Delta ABC = \Delta BCI + \Delta BIA + \Delta CIA$

$$\Delta = \frac{1}{2}(BC)(ID) + \frac{1}{2}(AB)(IF) + \frac{1}{2}(AC)(IE)$$

$$\Delta = \frac{1}{2}(a)(r) + \frac{1}{2}(c)(r) + \frac{1}{2}(b)(r)$$

$$\Delta = \frac{r}{2}[a+b+c] \qquad \because 2S = a+b+c$$

$$\Delta = \frac{r}{2}(2S)$$

$$\Delta = rS$$

$$\Rightarrow r = \frac{\Delta}{S} \quad \text{(Proved)}$$

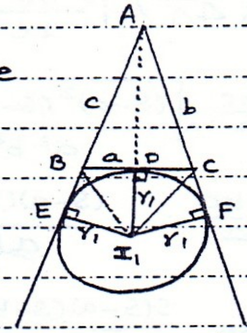
ESCRIBED CIRCLE

The circle which touches one side of a triangle externally and other two sides on extending internally is called Escribed or e-Circle. The centre of this Circle is called e-centre and radius is called e-radius denoted by r_1, r_2, r_3 opposite to A, B, C vertices respectively.

Show that

$$r_1 = \frac{\Delta}{S-a}, \quad r_2 = \frac{\Delta}{S-b}, \quad r_3 = \frac{\Delta}{S-c}$$

Proof:-



Let I_1 be e-centre. Draw ⊥lars from I_1 to each side. Join I_1 to each vertex of triangle ABC.

Now $I_1D = I_1E = I_1F = r_1$

$$\Delta ABC = \Delta ABI_1 + \Delta ACI_1 - \Delta BCI_1$$

$$\Delta = \frac{1}{2}(AB)(I_1E) + \frac{1}{2}(AC)(I_1F) - \frac{1}{2}(BC)(I_1D)$$

$$\Delta = \frac{1}{2}(c)(r_1) + \frac{1}{2}(b)(r_1) - \frac{1}{2}(a)(r_1)$$

$$\Delta = \frac{r_1}{2}[b+c-a]$$

$$\Delta = \frac{r_1}{2}[a+b+c-2a]$$

$$\because 2S = a+b+c$$

So $$\Delta = \frac{r_1}{2}[2S-2a]$$

$$\Delta = \frac{r_1}{2}2(S-a)$$

$$\Delta = r_1(S-a)$$

$$r_1 = \frac{\Delta}{S-a} \quad \text{(Proved)}$$

Similarly

$$r_2 = \frac{\Delta}{S-b}$$

and

$$r_3 = \frac{\Delta}{S-c}$$

EXERCISE: 12.8

Q.1 Show that

(i) $r = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$.

$$\text{RHS} = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$$

$$= 4\left(\frac{abc}{4\Delta}\right)\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-c)(s-a)}{ca}}\sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{(s-a)^2(s-b)^2(s-c)^2}{a^2b^2c^2}}$$

$$= \frac{\cancel{abc}}{\Delta}\cdot\frac{(s-a)(s-b)(s-c)}{\cancel{abc}}$$

$$= \frac{1}{\Delta}\cdot\frac{s(s-a)(s-b)(s-c)}{s}$$

$$\because \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \frac{1}{\Delta}\cdot\frac{\Delta^2}{s} = \frac{\Delta}{s}$$

$$= r = \text{LHS}.$$

$\Rightarrow$ LHS = RHS.

(ii) $s = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

$$\text{RHS} = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$= 4\left(\frac{abc}{4\Delta}\right)\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{s^3(s-a)(s-b)(s-c)}{a^2b^2c^2}}$$

$$= \frac{\cancel{abc}}{\Delta}\cdot\frac{s\sqrt{s(s-a)(s-b)(s-c)}}{\cancel{abc}}$$

$$= \frac{s}{\Delta}\cdot\Delta = s = \text{LHS}.$$

$\Rightarrow$ LHS = RHS.

Q.2 Show that

(i) $r = a\cdot\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2}$

$$\text{RHS} = a\cdot\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2}$$

$$= a\,\frac{\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$$

$$= \frac{a\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}}{\sqrt{\frac{s(s-a)}{bc}}}$$

$$= a\sqrt{\frac{(s-a)^2(s-b)(s-c)\,\cancel{bc}}{a^2\cancel{bc}\,s(s-a)}}$$

$$= \cancel{a}\,\frac{\sqrt{(s-a)(s-b)(s-c)}}{\cancel{a}\sqrt{s}}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{s}\sqrt{s}} = \frac{\Delta}{s}$$

$$= r = \text{LHS}. \quad (\text{Proved})$$

(ii) $r = b\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}\sec\frac{\beta}{2}$.

$$\text{RHS} = b\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}\sec\frac{\beta}{2}$$

$$= b\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{ac}{s(s-b)}}$$

$$= b\sqrt{\frac{(s-a)(s-b)^2(s-c)\,\cancel{ac}}{s(s-b)\,\cancel{a}b^2\cancel{c}}}$$

$$= \cancel{b}\cdot\frac{\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}\,\cancel{b}}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{s}\sqrt{s}} = \frac{\Delta}{s}$$

$$= r = \text{LHS} \quad (\text{Proved})$$

(iii) $r = c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}$

$$\text{RHS} = c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}$$

$$= c\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{ab}{s(s-c)}}$$

$$= c\sqrt{\frac{(s-a)(s-b)(s-c)^2\,\cancel{ab}}{s(s-c)\,\cancel{ab}c^2}}$$

$$= \cancel{c}\,\frac{\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}\,\cancel{c}}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{s}\sqrt{s}} = \frac{\Delta}{s}$$

$$= r = \text{LHS}. \quad (\text{Proved})$$

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Q.3/ Show that:

(i) $r_1 = 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

RHS $= 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

$= 4\left(\frac{abc}{4\Delta}\right)\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}$

$= \frac{abc}{\Delta}\sqrt{\frac{s^2(s-b)^2(s-c)^2}{a^2b^2c^2}}$

$= \frac{\cancel{abc}}{\Delta}\cdot\frac{s(s-b)(s-c)}{\cancel{abc}}$

$= \frac{s(s-a)(s-b)(s-c)}{\Delta(s-a)} = \frac{\Delta^2}{\Delta(s-a)}$

$= \frac{\Delta}{s-a} = r_1 =$ LHS. (Proved)

(ii) $r_2 = 4R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}$

RHS $= 4R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}$

$= 4\left(\frac{abc}{4\Delta}\right)\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{s(s-c)}{ab}}$

$= \frac{abc}{\Delta}\sqrt{\frac{s^2(s-a)^2(s-c)^2}{a^2b^2c^2}}$

$= \frac{\cancel{abc}}{\Delta}\cdot\frac{s(s-a)(s-c)}{\cancel{abc}}$

$= \frac{s(s-a)(s-b)(s-c)}{\Delta(s-b)} = \frac{\Delta^2}{\Delta(s-b)}$

$= \frac{\Delta}{s-b} = r_2 =$ LHS (Proved)

(iii) $r_3 = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}$

RHS $= 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}$

$= 4\left(\frac{abc}{4\Delta}\right)\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}$

$= \frac{abc}{\Delta}\sqrt{\frac{s^2(s-a)^2(s-b^2}{a^2b^2c^2}}$

$= \frac{\cancel{abc}}{\Delta}\cdot\frac{s(s-a)(s-b)}{\cancel{abc}}$

$= \frac{s(s-a)(s-b)(s-c)}{\Delta(s-c)} = \frac{\Delta^2}{\Delta(s-c)}$

$= \frac{\Delta}{s-c} = r_3 =$ LHS (Proved)

Q4 (i) $r_1 = s\tan\frac{\alpha}{2}$

RHS $= s\tan\frac{\alpha}{2}$

$= s\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

$= s\sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-a)^2}}$

$= s\cdot\frac{\Delta}{s(s-a)} = \frac{\Delta}{s-a}$

$= r_1 =$ LHS (Proved)

(ii) $r_2 = s\tan\frac{\beta}{2}$

RHS $= s\tan\frac{\beta}{2}$

$= s\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

$= s\sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-b)^2}}$

$= s\,\frac{\Delta}{s(s-b)} = \frac{\Delta}{s-b}$

$= r_2 =$ LHS (Proved)

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(iii) $r_3 = s\tan\frac{\gamma}{2}$

RHS $= s\tan\frac{\gamma}{2}$

$= s\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

$= s\sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-c)^2}}$

$= \frac{s\,\Delta}{s(s-c)} = \frac{\Delta}{s-c}$

$= r_3 =$ LHS (Proved)

Q.4 (ii) $r\,r_1\,r_2\,r_3 = \Delta^2$

LHS $= r\,r_1\,r_2\,r_3$

$= \frac{\Delta}{s}\cdot\frac{\Delta}{s-a}\cdot\frac{\Delta}{s-b}\cdot\frac{\Delta}{s-c}$

$= \frac{\Delta^4}{s(s-a)(s-b)(s-c)} = \frac{\Delta^4}{\Delta^2}$

$= \Delta^2 =$ RHS

$\Rightarrow$ LHS = RHS (Proved)

(i) $r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$

$LHS = r_1 r_2 + r_2 r_3 + r_3 r_1$

$= \frac{\Delta}{s-a}\cdot\frac{\Delta}{s-b} + \frac{\Delta}{s-b}\cdot\frac{\Delta}{s-c} + \frac{\Delta}{s-c}\cdot\frac{\Delta}{s-a}$

$= \frac{\Delta^2}{(s-a)(s-b)} + \frac{\Delta^2}{(s-b)(s-c)} + \frac{\Delta^2}{(s-c)(s-a)}$

$= \Delta^2\left[\frac{s-c+s-a+s-b}{(s-a)(s-b)(s-c)}\right]$

$= \Delta^2 s\left[\frac{3s-(a+b+c)}{s(s-a)(s-b)(s-c)}\right]$

$= \Delta^2 s\,\frac{(3s-2s)}{\Delta^2}$ $\quad \because (2s = a+b+c)$

$= s(s)$

$= s^2 = RHS$

$\Rightarrow LHS = RHS$ (Proved)

(iii) $r_1 + r_2 + r_3 - r = 4R.$

$LHS = r_1 + r_2 + r_3 - r$

$= \frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} - \frac{\Delta}{s}$

$= \Delta\left[\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s}\right]$

$= \Delta\left[\frac{s-b+s-a}{(s-a)(s-b)} + \frac{s-s+c}{s(s-c)}\right]$

$= \Delta\left[\frac{2s-b-a}{(s-a)(s-b)} + \frac{c}{s(s-c)}\right]$

$\because 2s = a+b+c$

$= \Delta\left[\frac{a+b+c-b-a}{(s-a)(s-b)} + \frac{c}{s(s-c)}\right]$

$= \Delta\left[\frac{c}{(s-a)(s-b)} + \frac{c}{s(s-c)}\right]$

$= \Delta c\left[\frac{1}{(s-a)(s-b)} + \frac{1}{s(s-c)}\right]$

$= \Delta c\left[\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right]$

$= \Delta c\left[\frac{s^2-sc+s^2-as-bs+ab}{\Delta^2}\right]$

$= c\left[\frac{2s^2-s(a+b+c)+ab}{\Delta}\right]$

$= c\left[\frac{2s^2-s(2s)+ab}{\Delta}\right]$

$= \frac{c}{\Delta}\left[\cancel{2s^2}-\cancel{2s^2}+ab\right]$

$= \frac{abc}{\Delta} = 4\left[\frac{abc}{4\Delta}\right]$

$= 4R = RHS$

$\Rightarrow LHS = RHS$ (Proved)

(iv) $r_1 r_2 r_3 = r s^2$

$LHS = r_1 r_2 r_3$

$= \frac{\Delta}{s-a}\cdot\frac{\Delta}{s-b}\cdot\frac{\Delta}{s-c}$

$= \frac{s\cdot\Delta^3}{s(s-a)(s-b)(s-c)}$

$= \frac{s\Delta^3}{\Delta^2} = \Delta s$ $\quad r = \frac{\Delta}{s}$

$= (rs)s$ $\quad \because \Delta = sr$

$= rs^2 = RHS$

$\Rightarrow LHS = RHS$ (Proved)

Q.6 Find R, r, r_1, r_2, r_3.

(i) $a = 13 \quad b = 14 \quad c = 15$

$\because s = \frac{a+b+c}{2} = \frac{13+14+15}{2} = \frac{42}{2} = 21$

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\Delta = \sqrt{21(21-13)(21-14)(21-15)}$

$\Delta = \sqrt{21\times 8\times 7\times 6} = \sqrt{7056}$

$\Delta = 84$

$R = \frac{abc}{4\Delta} = \frac{(13)(14)(15)}{4(84)} = 8.125$

$r = \frac{\Delta}{s} = \frac{84}{21} = 4$

$r_1 = \frac{\Delta}{s-a} = \frac{84}{8} = 10.5$

$r_2 = \frac{\Delta}{s-b} = \frac{84}{7} = 12$

$r_3 = \frac{\Delta}{s-c} = \frac{84}{6} = 14$

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(ii) $a = 34 \quad b = 20 \quad c = 42$

$S = \frac{a+b+c}{2} = \frac{34+20+42}{2} = 48$

$\Delta = \sqrt{S(S-a)(S-b)(S-c)}$

$\Delta = \sqrt{48(48-34)(48-20)(48-42)}$

$\Delta = \sqrt{48 \times 14 \times 28 \times 6} = 336$

$R = \frac{abc}{4\Delta} = \frac{34 \times 20 \times 42}{4 \times 336} = 21.25$

$r = \frac{\Delta}{S} = \frac{336}{48} = 7$

$r_1 = \frac{\Delta}{S-a} = \frac{336}{14} = 24$

$r_2 = \frac{\Delta}{S-b} = \frac{336}{28} = 12$

$r_3 = \frac{\Delta}{S-c} = \frac{336}{6} = 56$

Q.7 Show that in equilateral triangle:

(i) $r : R : r_1 = 1 : 2 : 3$

$\because \Delta$ is equilateral so $a = b = c$

$S = \frac{a+a+a}{2} = \frac{3a}{2}$

$\Delta = \sqrt{S(S-a)(S-a)(S-a)} = \sqrt{S(S-a)^3}$

$= \sqrt{\frac{3a}{2}\left(\frac{3a}{2} - a\right)^3} = \sqrt{\frac{3a}{2} \cdot \left(\frac{a}{2}\right)^3}$

$= \sqrt{\frac{3a^4}{16}} = \frac{\sqrt{3}a^2}{4}$

$R = \frac{a \cdot a \cdot a}{4\left(\frac{\sqrt{3}a^2}{4}\right)} = \frac{a}{\sqrt{3}}$

$r = \frac{\Delta}{S} = \frac{\frac{\sqrt{3}a^2}{4}}{\frac{3a}{2}} = \frac{\sqrt{3}a^2}{4} \times \frac{2}{3a} = \frac{a}{2\sqrt{3}}$

$r_1 = r_2 = r_3 = \frac{\Delta}{S-a} = \frac{\frac{\sqrt{3}a^2}{4}}{a/2} = \frac{\sqrt{3}a}{2}$

Now

$r : R : r_1 = \frac{a}{2\sqrt{3}} : \frac{a}{\sqrt{3}} : \frac{\sqrt{3}a}{2}$

Multiplying by $\frac{2\sqrt{3}}{a}$

$= \frac{\cancel{a}}{\cancel{2\sqrt{3}}} \times \frac{\cancel{2\sqrt{3}}}{\cancel{a}} : \frac{\cancel{a}}{\cancel{\sqrt{3}}} \times \frac{2\cancel{\sqrt{3}}}{\cancel{a}} : \frac{\sqrt{3}\cancel{a}}{\cancel{2}} \times \frac{\cancel{2}\sqrt{3}}{\cancel{a}}$

$\Rightarrow r : R : r_1 = 1 : 2 : 3$ (Proved)

(ii) $r : R : r_1 : r_2 : r_3$

$= \frac{a}{2\sqrt{3}} : \frac{a}{\sqrt{3}} : \frac{\sqrt{3}a}{2} : \frac{\sqrt{3}a}{2} : \frac{\sqrt{3}a}{2}$

Multiplying by $\frac{2\sqrt{3}}{a}$

$= \frac{\cancel{a}}{\cancel{2\sqrt{3}}} \times \frac{\cancel{2\sqrt{3}}}{\cancel{a}} : \frac{\cancel{a}}{\cancel{\sqrt{3}}} \times \frac{2\cancel{\sqrt{3}}}{\cancel{a}} : \frac{\sqrt{3}\cancel{a}}{\cancel{2}} \times \frac{\cancel{2}\sqrt{3}}{\cancel{a}} : \frac{\sqrt{3}\cancel{a}}{\cancel{2}} \times \frac{\cancel{2}\sqrt{3}}{\cancel{a}} : \frac{\sqrt{3}\cancel{a}}{\cancel{2}} \times \frac{\cancel{2}\sqrt{3}}{\cancel{a}}$

$= 1 : 2 : 3 : 3 : 3$

Thus $r : R : r_1 : r_2 : r_3 = 1 : 2 : 3 : 3 : 3$ (Proved)

Q.8 Prove that:

(i) $\Delta = r^2 \operatorname{Cot}\frac{\alpha}{2} \cdot \operatorname{Cot}\beta/2 \cdot \operatorname{Cot}\frac{\gamma}{2}$

$RHS = r^2 \sqrt{\frac{S(S-a)}{(S-b)(S-c)}} \sqrt{\frac{S(S-b)}{(S-a)(S-c)}} \sqrt{\frac{S(S-c)}{(S-a)(S-b)}}$

$= r^2 \sqrt{\frac{S^3(S-a)(S-b)(S-c)}{(S-a)^2 \cdot (S-b)^2 \cdot (S-c)^2}}$

$= r^2 \sqrt{S^3} \sqrt{\frac{S}{S(S-a)(S-b)(S-c)}}$

$\because r = \frac{\Delta}{S}$

$= \frac{r^2 S^2}{\Delta} = \frac{\Delta^2}{S^2} \cdot \frac{S^2}{\Delta} = \Delta$

$= \Delta = LHS$

$\Rightarrow LHS = RHS$ (Proved)

(ii) $r = S \tan\frac{\alpha}{2} \cdot \tan\frac{\beta}{2} \cdot \tan\frac{\gamma}{2}$

$RHS = S \tan\frac{\alpha}{2} \tan\beta/2 \tan\gamma/2$

$= S \sqrt{\frac{(S-b)(S-c)}{S(S-a)}} \sqrt{\frac{(S-a)(S-c)}{S(S-b)}} \sqrt{\frac{(S-a)(S-b)}{S(S-c)}}$

$= S \sqrt{\frac{(S-a)^2(S-b)^2(S-c)^2}{S^3(S-a)(S-b)(S-c)}}$

$= S \frac{\sqrt{(S-a)(S-b)(S-c)}}{\sqrt{S^3}} \times \frac{\sqrt{S}}{\sqrt{S}}$

$= \frac{S}{S^2} \sqrt{S(S-a)(S-b)(S-c)} = \frac{1}{S} \Delta$

$= \frac{\Delta}{S} = r = LHS$

$\Rightarrow LHS = RHS$ (Proved)

(iii) $\Delta = 4R\gamma \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

$RHS = 4Rr\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

$= 4\left(\frac{abc}{4\Delta}\right)\left(\frac{\Delta}{s}\right)\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}$

$= \frac{abc}{s}\sqrt{\frac{s^3(s-a)(s-b)(s-c)}{a^2b^2c^2}}$

$= \frac{\cancel{abc}}{\cancel{s}}\cdot \cancel{s}\frac{\Delta}{\cancel{abc}} = \Delta$

$= \Delta = LHS$

$\Rightarrow LHS = RHS$ (Proved)

Q.9 Show that:

(i) $\frac{1}{2R\gamma} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$

$RHS = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$

$= \frac{c+a+b}{abc}$ $\quad \because 2s = a+b+c$

$= \frac{2s}{abc} = \frac{2(\Delta/r)}{4\Delta R}$ $\quad \because \frac{\Delta}{s} = \gamma \Rightarrow s = \frac{\Delta}{r}$

$= \frac{2\Delta}{4\Delta R\gamma}$ $\quad \because R = \frac{abc}{4\Delta} \Rightarrow abc = 4\Delta R$

$= \frac{1}{2\gamma R} = LHS$

$\Rightarrow LHS = RHS$ (Proved)

(ii) $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

$RHS = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

$= \frac{1}{(\Delta/s-a)} + \frac{1}{(\Delta/s-b)} + \frac{1}{(\Delta/s-c)}$

$= \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta}$

$= \frac{s-a+s-b+s-c}{\Delta}$

$= \frac{3s-(a+b+c)}{\Delta}$

$= \frac{3s-2s}{\Delta}$ $\quad \because 2s = a+b+c$

$= \frac{s}{\Delta} = \frac{1}{(\Delta/s)} = \frac{1}{r}$

$= LHS$

$\Rightarrow LHS = RHS$ (Proved)

Q.10 Prove that:

(i) $r = \frac{a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$

$RHS = \frac{a\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}}{\sqrt{\frac{s(s-a)}{bc}}}$

$= a\sqrt{\frac{(s-a)^2(s-c)(s-b)\cancel{bc}}{s(s-a)\,a^2\cancel{bc}}}$

$= \cancel{a}\,\frac{\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}\,\cancel{a}}$

$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{s}\cdot\sqrt{s}} = \frac{\Delta}{s}$

$= \gamma = LHS$ (Proved)

(ii) $r = \frac{b\sin\frac{\alpha}{2}\cdot\sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}}$

$RHS = \frac{b\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}}{\cos\frac{\beta}{2}}$

$= \frac{b\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-b)}{ab}}}{\sqrt{\frac{s(s-b)}{ac}}}$

$= b\sqrt{\frac{(s-a)(s-b)^2(s-c)\cancel{ac}}{s(s-b)\,\cancel{ac}\,b^2}}$

$= \cancel{b}\,\frac{\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}\,\cancel{b}}$

$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{s}\sqrt{s}} = \frac{\Delta}{s}$

$= r = LHS$ (Proved)

(iii) $r = \frac{c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$

$RHS = \frac{c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$

$= \frac{c\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}}{\sqrt{\frac{s(s-c)}{ab}}}$

$$= c\sqrt{\frac{(s-a)(s-b)(s-c)^2\cancel{ab}}{s(s-c)\cancel{ab}c^2}}$$

$$= \frac{\cancel{c}\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}\ \cancel{c}}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{s}\sqrt{s}} = \frac{\Delta}{s}$$

$$= r = LHS \text{ (Proved)}$$

Q.11 Show that:

$$abc(\sin\alpha+\sin\beta+\sin\gamma) = 4\Delta s.$$

$$LHS = abc(\sin\alpha+\sin\beta+\sin\gamma)$$

$$\because 2R = \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

$$\Rightarrow \sin\alpha = \frac{a}{2R},\ \sin\beta = \frac{b}{2R},\ \sin\gamma = \frac{c}{2R}$$

$$= abc\left[\frac{a}{2R}+\frac{b}{2R}+\frac{c}{2R}\right]$$

$$= \frac{abc}{2R}[a+b+c] \qquad \because 2s = a+b+c$$

$$= \frac{\cancel{abc}}{\cancel{2}\left(\frac{\cancel{abc}}{4\Delta}\right)}\ \cancel{2}s$$

$$= 4\Delta s = RHS$$

$$\Rightarrow LHS = RHS \text{ (Proved)}$$

Q.12 Prove that:

(i) $(r_1+r_2)\tan\frac{\gamma}{2} = c$

$$LHS = (r_1+r_2)\tan\gamma/2$$

$$= \left[\frac{\Delta}{s-a}+\frac{\Delta}{s-b}\right]\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \Delta\left[\frac{1}{s-a}+\frac{1}{s-b}\right]\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \Delta\left[\frac{s-b+s-a}{(s-a)(s-b)}\right]\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$\because 2s = a+b+c$$

$$= \Delta\left[\frac{a+b+c-a-b}{(s-a)(s-b)}\right]\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \frac{\Delta c}{\sqrt{s(s-a)(s-b)(s-c)}} \qquad \because \frac{\sqrt{a}}{a} = \frac{1}{\sqrt{a}}$$

$$= \frac{\Delta c}{\Delta} = c = RHS$$

$$\Rightarrow LHS = RHS \text{ (Proved)}$$

(ii) $(r_3-r)\cot\frac{\gamma}{2} = c$

$$LHS = (r_3-r)\cot\gamma/2$$

$$= \left[\frac{\Delta}{s-c}-\frac{\Delta}{s}\right]\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \Delta\left[\frac{\cancel{s}-\cancel{s}+c}{s(s-c)}\right]\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \frac{\Delta c}{\sqrt{s(s-a)(s-b)(s-c)}} \qquad \because \frac{\sqrt{a}}{a} = \frac{1}{\sqrt{a}}$$

$$= \frac{\Delta c}{\Delta} = c = RHS$$

$$\Rightarrow LHS = RHS \text{ (Proved)}$$

The End

Example 3 - Prove that

$$\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2} = \frac{a^2+b^2+c^2}{\Delta^2}$$

$$LHS = \frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}$$

$$= \frac{s^2}{\Delta^2}+\frac{(s-a)^2}{\Delta^2}+\frac{(s-b)^2}{\Delta^2}+\frac{(s-c)^2}{\Delta^2}$$

$$= \frac{1}{\Delta^2}\left[s^2+(s-a)^2+(s-b)^2+(s-c)^2\right]$$

$$= \frac{1}{\Delta^2}\left[s^2+s^2+a^2-2as+s^2+b^2-2bs+s^2+c^2-2cs\right]$$

$$= \frac{1}{\Delta^2}\left[4s^2 - 2s(a+b+c) + a^2+b^2+c^2\right]$$

$$= \frac{1}{\Delta^2}\left[4s^2 - 2s(2s) + a^2+b^2+c^2\right]$$

$\because 2s = a+b+c$

$$= \frac{1}{\Delta^2}\left[\cancel{4s^2} - \cancel{4s^2} + a^2+b^2+c^2\right]$$

$$= \frac{a^2+b^2+c^2}{\Delta^2} = RHS$$

$\Rightarrow$ LHS = RHS (Proved)

Example ①: Show that:

$$r = (s-a)\tan\frac{\alpha}{2}$$

$$RHS = (s-a)\tan\frac{\alpha}{2}$$

$$= (s-a)\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= \frac{\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}} \qquad \left(\because \frac{a}{\sqrt{a}} = \sqrt{a}\right)$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{s}\sqrt{s}}$$

$$= \frac{\Delta}{s} = r = LHS$$

$\Rightarrow$ LHS = RHS (Proved)

Example ② Show that

$$r_1 = 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$RHS = 4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$= 4\left(\frac{abc}{4\Delta}\right)\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}$$

$$= \frac{abc}{\Delta}\sqrt{\frac{s^2(s-b)^2(s-c)^2}{a^2b^2c^2}}$$

$$= \frac{\cancel{abc}}{\Delta}\cdot\frac{s(s-b)(s-c)}{\cancel{abc}}$$

$$= \frac{s(s-a)(s-b)(s-c)}{\Delta(s-a)}$$

$$= \frac{\Delta^2}{\Delta(s-a)}$$

$$= \frac{\Delta}{s-a} = r_1 = LHS$$

$\Rightarrow$ LHS = RHS (Proved)

Summary

$$\sin\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \operatorname{cosec}\frac{\alpha}{2} = \sqrt{\frac{bc}{(s-b)(s-c)}}$$

$$\sin\frac{\beta}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}, \quad \operatorname{cosec}\frac{\beta}{2} = \sqrt{\frac{ac}{(s-a)(s-c)}}$$

$$\sin\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}, \quad \operatorname{cosec}\frac{\gamma}{2} = \sqrt{\frac{ab}{(s-a)(s-b)}}$$

$$\cos\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}, \quad \sec\frac{\alpha}{2} = \sqrt{\frac{bc}{s(s-a)}}$$

$$\cos\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{ac}}, \quad \sec\frac{\beta}{2} = \sqrt{\frac{ac}{s(s-b)}}$$

$$\cos\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{ab}}, \quad \sec\frac{\gamma}{2} = \sqrt{\frac{ab}{s(s-c)}}$$

$$\tan\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \quad \cot\frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$\tan\frac{\beta}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \quad \cot\frac{\beta}{2} = \sqrt{\frac{s(s-b)}{(s-a)(s-c)}}$$

$$\tan\frac{\gamma}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}, \quad \cot\frac{\gamma}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$R = \frac{abc}{4\Delta}, \quad r = \frac{\Delta}{s}, \quad r_1 = \frac{\Delta}{s-a}$$

$$r_2 = \frac{\Delta}{s-b}, \quad r_3 = \frac{\Delta}{s-c}$$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$2R = \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

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