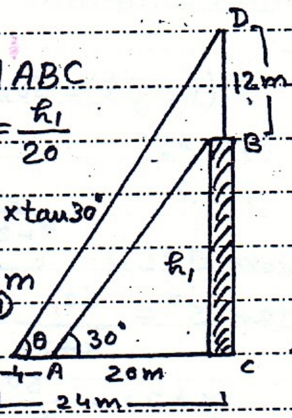
Q.14



From ΔABC

$$\tan 30^\circ = \frac{h_1}{20}$$

$$h_1 = 20 \times \tan 30^\circ$$

$$h_1 = 11.55\,m \quad \text{——}(1)$$

Now from ΔECD

$$\tan\theta = \frac{h_1 + 12}{20+4}$$

$$\tan\theta = \frac{11.55+12}{24}$$

$$\tan\theta = \frac{23.55}{24}$$

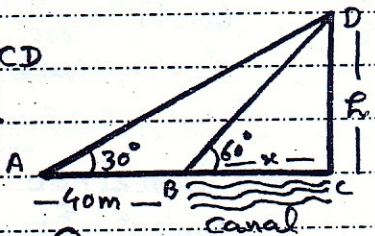
$$\tan\theta = 0.98125$$

$$\theta = \tan^{-1}(0.98125)$$

$$\theta = 44^\circ 27' 15''$$

New angle of elevation is $44^\circ 27' 15''$.

Q.15



From ΔBCD

$$\tan 60^\circ = \frac{h}{x}$$

$$\sqrt{3} = \frac{h}{x}$$

$$h = \sqrt{3}x \quad \text{——}(1)$$

Now from ΔACD

$$\tan 30^\circ = \frac{h}{x+40}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{x+40}$$

$$x+40 = \sqrt{3}h$$

$$x+40 = \sqrt{3}(\sqrt{3}x)$$

$$x+40 = 3x$$

$$40 = 3x - x = 2x$$

$$\boxed{x = 20m}$$

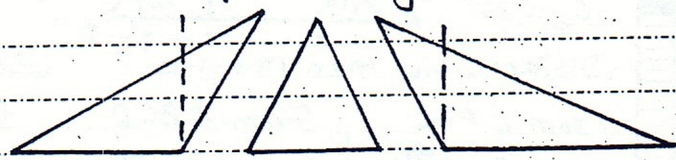
Now

$$h = \sqrt{3} \times 20 = 34.64$$

$$\boxed{h = 34.64\,m}$$

Oblique Triangle:-

"A triangle which is not right triangle is called oblique triangle".



Solution of Oblique triangle:-

Oblique triangles can be solved by the following Laws:

v.v. Imp.

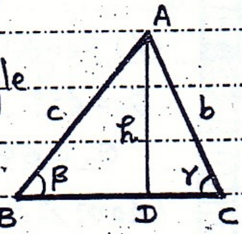
Law of Sines:-

In any triangle ABC

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

is called Law of Sines.

Proof:-



In oblique triangle ABC, Draw $AD \perp BC$ where $|AD| = h$

From ΔABD and ΔADC

$$\sin\beta = \frac{h}{c} \qquad \sin\gamma = \frac{h}{b}$$

$$h = c\sin\beta \qquad h = b\sin\gamma$$

$$\Rightarrow c\sin\beta = b\sin\gamma$$

$$\Rightarrow \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \quad \text{——}(A)$$

Similarly by drawing $\perp$ from B to AC and C to AB

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma} \text{——}(B) \quad \text{and} \quad \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \text{——}(C)$$

From (A), (B), (C)

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

which is Law of Sines.

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Law of Cosines:-

In any triangle ABC

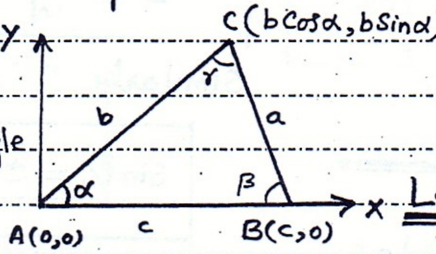
$a^2 = b^2 + c^2 - 2bc\,\cos\alpha$

$b^2 = c^2 + a^2 - 2ca\,\cos\beta$

$c^2 = a^2 + b^2 - 2ab\,\cos\gamma$

are called Laws of Cosines.

Proof:- Consider an oblique triangle with vertices $A(0,0)$, $B(c,0)$, $C(b\cos\alpha, b\sin\alpha)$



Using Distance Formula:

$|BC| = \sqrt{(b\cos\alpha - c)^2 + (b\sin\alpha - 0)^2}$

$a = \sqrt{b^2\cos^2\alpha + c^2 - bc\cos\alpha(2) + b^2\sin^2\alpha}$

Squaring both Sides:

$a^2 = b^2(\cos^2\alpha + \sin^2\alpha) + c^2 - 2bc\cos\alpha$

$\because \sin^2\alpha + \cos^2\alpha = 1$

$\boxed{a^2 = b^2 + c^2 - 2bc\cos\alpha}$ (Proved)

Similarly

$\boxed{b^2 = c^2 + a^2 - 2ca\cos\beta}$

$\boxed{c^2 = a^2 + b^2 - 2ab\cos\gamma}$

Now $a^2 = b^2 + c^2 - 2bc\cos\alpha$

$2bc\cos\alpha = b^2 + c^2 - a^2$

$$\boxed{\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}}$$

Similarly

$$\boxed{\cos\beta = \frac{c^2 + a^2 - b^2}{2ca}}$$

$$\boxed{\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}}$$

Deduction of Pythagorous Theorem

$\because a^2 = b^2 + c^2 - 2bc\cos\alpha$

If $\alpha = 90^\circ \Rightarrow \cos\alpha = 0$ so

$\boxed{a^2 = b^2 + c^2}$

Similarly

$b^2 = c^2 + a^2$ if $\beta = 90^\circ$

$c^2 = a^2 + b^2$ if $\gamma = 90^\circ$

Law of Tangents:-

In any triangle ABC

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$$

$$\frac{b-c}{b+c} = \frac{\tan\left(\frac{\beta-\gamma}{2}\right)}{\tan\left(\frac{\beta+\gamma}{2}\right)}$$

$$\frac{c-a}{c+a} = \frac{\tan\left(\frac{\gamma-\alpha}{2}\right)}{\tan\left(\frac{\gamma+\alpha}{2}\right)}$$

are called Laws of tangents.

Proof:- Consider

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = K \text{ (let)}$$

$\frac{a}{\sin\alpha} = K$ and $\frac{b}{\sin\beta} = K$

$a = K\sin\alpha \qquad b = K\sin\beta$

$$\frac{a-b}{a+b} = \frac{K\sin\alpha - K\sin\beta}{K\sin\alpha + K\sin\beta}$$

$$\frac{a-b}{a+b} = \frac{K(\sin\alpha - \sin\beta)}{K(\sin\alpha + \sin\beta)}$$

$$\frac{a-b}{a+b} = \frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}$$

$\because \sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

$\sin P - \sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

$$\frac{a-b}{a+b} = \cot\left(\frac{\alpha+\beta}{2}\right)\tan\left(\frac{\alpha-\beta}{2}\right)$$

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{\alpha-\beta}{2}\right)}{\tan\left(\frac{\alpha+\beta}{2}\right)}$$

Similarly

$$\frac{b-c}{b+c} = \frac{\tan\left(\frac{\beta-\gamma}{2}\right)}{\tan\left(\frac{\beta+\gamma}{2}\right)}$$

$$\frac{c-a}{c+a} = \frac{\tan\left(\frac{\gamma-\alpha}{2}\right)}{\tan\left(\frac{\gamma+\alpha}{2}\right)}$$

Hero's Formula:-

If a, b, c are the Sides of a triangle ABC then the relation

$$2S = a+b+c$$

$$S = \frac{a+b+c}{2}$$

are called Hero's Formula. "S" is called Semi parameter of triangle ABC.

Half Angle Formulas:-

(a) Sine Half Angle Formulae:-

We know that

$$\cos\alpha = 1 - 2\sin^2 \alpha/2$$

$$\Rightarrow 2\sin^2\frac{\alpha}{2} = 1 - \cos\alpha$$

$\because \cos\alpha = \frac{b^2+c^2-a^2}{2bc}$

$$2\sin^2\frac{\alpha}{2} = 1 - \left(\frac{b^2+c^2-a^2}{2bc}\right)$$

$$= \frac{2bc-b^2-c^2+a^2}{2bc}$$

$$= \frac{a^2-(b^2+c^2-2bc)}{2bc}$$

$$\sin^2\frac{\alpha}{2} = \frac{a^2-(b-c)^2}{4bc}$$

$$\sin^2\frac{\alpha}{2} = \frac{(a+b-c)(a-b+c)}{4bc}$$

$$\sin^2\frac{\alpha}{2} = \frac{(a+b+c-2c)(a+b+c-2b)}{4bc}$$

$$\sin^2\frac{\alpha}{2} = \frac{(2S-2b)(2S-2c)}{4bc}$$

$\because 2S = a+b+c$

$$\sin^2\frac{\alpha}{2} = \frac{4(S-b)(S-c)}{4bc}$$

$$\sin^2\frac{\alpha}{2} = \frac{(S-b)(S-c)}{bc}$$

$$\boxed{\sin\frac{\alpha}{2} = \pm\sqrt{\frac{(S-b)(S-c)}{bc}}}$$

Similarly

$$\boxed{\sin \beta/2 = \pm\sqrt{\frac{(S-a)(S-c)}{ac}}}$$

$$\boxed{\sin\frac{\gamma}{2} = \pm\sqrt{\frac{(S-a)(S-b)}{ab}}}$$

(b) Cosine Half Angle Formulae:-

We know that

$$\cos\alpha = 2\cos^2 \alpha/2 - 1$$

$$2\cos^2\frac{\alpha}{2} = 1 + \cos\alpha$$

$\because \cos\alpha = \frac{b^2+c^2-a^2}{2bc}$

$$2\cos^2\frac{\alpha}{2} = 1 + \left(\frac{b^2+c^2-a^2}{2bc}\right)$$

$$2\cos^2\frac{\alpha}{2} = \frac{2bc+b^2+c^2-a^2}{2bc}$$

$$\cos^2\frac{\alpha}{2} = \frac{(b+c)^2-a^2}{4bc}$$

$$\cos^2\frac{\alpha}{2} = \frac{(b+c+a)(b+c-a)}{4bc}$$

$$\cos^2\frac{\alpha}{2} = \frac{2S(a+b+c-2a)}{4bc}$$

$$\cos^2\frac{\alpha}{2} = \frac{2S(2S-2a)}{4bc}$$

$$\boxed{\cos\frac{\alpha}{2} = \pm\sqrt{\frac{S(S-a)}{bc}}}$$

Similarly

$$\boxed{\cos \beta/2 = \pm\sqrt{\frac{S(S-b)}{ac}}}, \quad \boxed{\cos\frac{\gamma}{2} = \pm\sqrt{\frac{S(S-c)}{ab}}}$$

(C) Tangents half angle Formulae:-

$$\because \tan\frac{\alpha}{2} = \frac{\sin\alpha/2}{\cos\alpha/2}$$

$$\tan\frac{\alpha}{2} = \pm\frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}}$$

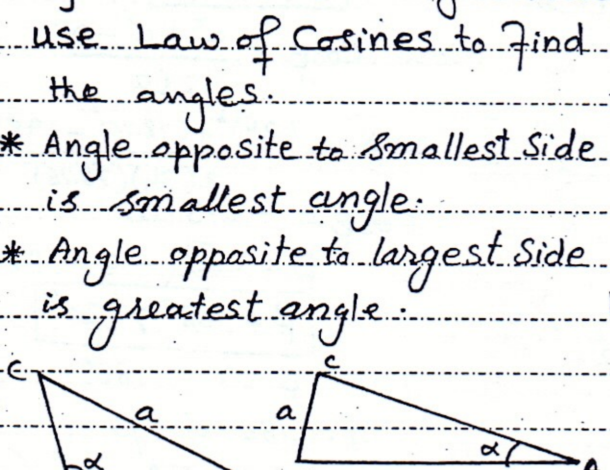
$$\boxed{\tan\frac{\alpha}{2} = \pm\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}}$$

Similarly

$$\boxed{\tan\beta/2 = \pm\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}}$$

$$\boxed{\tan\frac{\gamma}{2} = \pm\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}}$$

* If two Sides and third angle is given then use Law of Cosines.
* If two Sides and one angle of Corresponding Sides OR one side and two angles are given then Use Law of Sines.
* If all 3 sides are given then use Law of Cosines to find the angles.
* Angle opposite to Smallest Side is smallest angle.
* Angle opposite to largest Side is greatest angle.



EXERCISE: 12.4

Solve the triangles ABC:-

Q.1 $a = ?$ $b = \sqrt{6}$ $c = ?$

$\alpha = ?$ $\beta = 60^\circ$ $\gamma = 15^\circ$

Using $\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$

$$c = \frac{b}{\sin\beta} \times \sin\gamma = \frac{\sqrt{6}}{\sin 60} \times \sin 15^\circ$$

$$c = \frac{2.4495 \times 0.2588}{0.8660} = 0.732$$

$\boxed{c = \sqrt{3}-1}$ $\boxed{\because \sqrt{3} = 1.732}$

Now $\alpha + \beta + \gamma = 180^\circ$

$\alpha = 180^\circ - 60^\circ - 15^\circ = 105^\circ$

$\boxed{\alpha = 105^\circ}$

Now $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$

$$a = \frac{b}{\sin\beta} \times \sin\alpha = \frac{\sqrt{6}}{\sin 60^\circ} \times \sin 105^\circ$$

$$a = \frac{2.4495 \times 0.9659}{0.8660} = 2.732$$

$\boxed{a = \sqrt{3}+1}$ $\boxed{\because \sqrt{3} = 1.732}$

Q.2 $a = 89.35$ $b = ?$ $c = ?$

$\alpha = ?$ $\beta = 52^\circ$ $\gamma = 89^\circ 35'$

$\because \alpha + \beta + \gamma = 180^\circ$

$\alpha = 180^\circ - \beta - \gamma$

$\alpha = 180^\circ - 52^\circ - 89^\circ 35'$

$\boxed{\alpha = 38^\circ 25'}$

Using $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$

$$b = \frac{a}{\sin\alpha} \times \sin\beta = \frac{89.35}{\sin 38^\circ 25'} \times \sin 52^\circ$$

$\boxed{b = 113.31}$

Again $\dfrac{a}{\sin\alpha} = \dfrac{c}{\sin\gamma}$

$c = \dfrac{a}{\sin\alpha} \times \sin\gamma = \dfrac{89.35}{\sin 38^\circ 25'} \times \sin 89^\circ 35'$

$\boxed{c = 143.79}$

Q.3 $a = ?$ $b = 125$ $c = ?$

$\alpha = 47^\circ$ $\beta = ?$ $\gamma = 53^\circ$

$\because \alpha + \beta + \gamma = 180^\circ$

$\beta = 180^\circ - \alpha - \gamma^\circ$

$\beta = 180^\circ - 47^\circ - 53^\circ$

$\boxed{\beta = 80^\circ}$

Using $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$

$\Rightarrow a = \dfrac{b}{\sin\beta} \times \sin\alpha = \dfrac{125}{\sin 80^\circ} \times \sin 47^\circ$

$\boxed{a = 92.83}$

Now $\dfrac{c}{\sin\gamma} = \dfrac{b}{\sin\beta}$

$c = \dfrac{b}{\sin\beta} \times \sin\gamma = \dfrac{125}{\sin 80^\circ} \times \sin 53^\circ$

$\boxed{c = 101.37}$

Q.4 $a = ?$ $b = ?$ $c = 16.1$

$\alpha = 42^\circ 45'$ $\beta = ?$ $\gamma = 74^\circ 32'$

$\because \alpha + \beta + \gamma = 180^\circ$

$\beta = 180^\circ - \alpha - \gamma$

$\beta = 180^\circ - 42^\circ 45' - 74^\circ 32'$

$\boxed{\beta = 62^\circ 43'}$

Using $\dfrac{a}{\sin\alpha} = \dfrac{c}{\sin\gamma}$

$a = \dfrac{c}{\sin\gamma} \times \sin\alpha = \dfrac{16.1}{\sin 74^\circ 32'} \times \sin 42^\circ 45'$

$\boxed{a = 11.34}$

Again $\dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$

$b = \dfrac{c}{\sin\gamma} \times \sin\beta = \dfrac{16.1}{\sin 74^\circ 32'} \times \sin 62^\circ 43'$

$\boxed{b = 14.85}$

Q.5 $a = 53$ $b = ?$ $c = ?$

$\alpha = ?$ $\beta = 88^\circ 36'$ $\gamma = 31^\circ 54'$

$\because \alpha + \beta + \gamma = 180^\circ$

$\alpha = 180^\circ - \beta - \gamma$

$\alpha = 180^\circ - 88^\circ 36' - 31^\circ 54'$

$\boxed{\alpha = 59^\circ 30'}$

Using $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$

$b = \dfrac{a}{\sin\alpha} \times \sin\beta = \dfrac{53}{\sin 59^\circ 30'} \times \sin 88^\circ 36'$

$\boxed{b = 61.51}$

Again $\dfrac{a}{\sin\alpha} = \dfrac{c}{\sin\gamma}$

$c = \dfrac{a}{\sin\alpha} \times \sin\gamma = \dfrac{53}{\sin 59^\circ 30'} \times \sin 31^\circ 54'$

$\boxed{c = 32.51}$

"EXERCISE: 12.5"

Q.1 $a = ?$ $b = 95$ $c = 34$

$\alpha = 52^\circ$ $\beta = ?$ $\gamma = ?$

Using $a^2 = b^2 + c^2 - 2bc\cos\alpha$

$a^2 = (95)^2 + (34)^2 - 2(95)(34)\cos 52^\circ$

$a^2 = 6203.83$

$\boxed{a = 78.76}$

Now $\cos\beta = \dfrac{c^2 + a^2 - b^2}{2ca}$

$\cos\beta = \dfrac{(34)^2 + (78.76)^2 - (95)^2}{2(34)(78.76)}$

$\cos\beta = -0.3110$

$\beta = \cos^{-1}(-0.3110)$

$\boxed{\beta = 108^\circ 7' 20''}$

$\because \alpha + \beta + \gamma = 180^\circ$

$\gamma = 180 - \alpha - \beta$

$\gamma = 180 - 52^\circ - 108^\circ 7' 20''$

$\boxed{\gamma = 19^\circ 52' 40''}$
